On (ψ, ϕ) Contraction in Bicomplex Valued Fuzzy b-Metric Spaces with Application

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Abstract

In this article, we introduce the concept of bicomplex valued fuzzy b-metric spaces and a modified (ϕ, ψ) fuzzy contraction. We construct certain fixed point results in bicomplex valued fuzzy b-metric spaces. Our work is inspired by I. Demir [1] and Singh *et al.* [2]. Some examples are provided to validate our results. Further, we substantiate the utility of our work in identifying the unique solution to a system of equations emerging in dynamic programming.

Index Terms

Bicomplex valued fuzzy b-metric space, Bicomplex valued metric space, Fixed point, , Modified (ϕ, ψ) fuzzy contraction.

I. Introduction

SEGRE [3] created a pioneering attempt in the development of special algebras. He conceptualized commutative generalization of complex numbers as bicomplex numbers, tricomplex numbers, etc. Subsequently the bicomplex algebra and function theory developed by Price [4]. Recently revived interest during this subject finds some prominent applications in different fields of mathematical sciences and different branches of science and technology. An efficacious body of work has been developed by several researchers.

S. Banach [5] provided a crucial result to the fixed point theory, the 'Banach contraction principle,' in 1922. In many disciplines of mathematical analysis, this principle is a prevalent and effective technique for solving existence problems, and is an active area of research.

On the other hand, important theoretical development in the fuzzy sets theory introduced by Zadeh [6]. Fuzzy sets theory is the way of defining the concept of fuzzy metric spaces by Kramosil and Michalek [7], which can be regarded as a generalization of the statistical metric spaces. Subsequently, M. Grabiec [8] defined G-complete fuzzy metric spaces and extended the complete fuzzy metric spaces. Following Grabiec's work, George and Veeramani [9] modified the notion of M-complete fuzzy metric spaces with the help of continuous t-norms. Many authors ([10], [11], [12], [13]) established and extended the numerous types of fuzzy contractive mappings, as well as explored some fixed point theorems in fuzzy metric spaces.

Buckley ([14] -[17]) was the first to present fuzzy complex numbers and fuzzy complex analysis. Some authors continued their research in fuzzy complex numbers after acknowledging Buckley's work. Ramot *et al.* [18] expanded fuzzy sets to complex fuzzy sets in this series. Singh *et al.* [2] utilized Ramot *et al.* [18] the thought of complex fuzzy sets and used continuous t-norms to define the concept of complex valued fuzzy metric spaces.

I. Demir [1] proposed the notion of a complex valued fuzzy b-metric spaces and investigated several fixed point theorems, based on the work of Shukla *et al.* [12].

II. Preliminaries

Now, we begin with some basic fundamental aspects, notations and definitions. Let $\mathbb{R}$ represent the set of real numbers, $\mathbb{R}_+$ represent the set of all non-negative real numbers and $\mathbb{N}$ represent the set of natural numbers. We start with the following definitions of a fuzzy metric space.

Definition II.1. *[9] An ordered triple $(X, M, *)$ is called fuzzy metric space such that X is a nonempty set, $*$ defined a continuous t-norm and M be a fuzzy set on $X \times X \times (0, \infty)$, satisfying the following conditions, for all $x, y, z \in X, s, t > 0$.*
(FM-1) $M(x, y, t) > 0$.
(FM-2) $M(x, y, t) = 1$ *iff* $x = y$.
(FM-3) $M(x, y, t) = M(y, x, t)$.
(FM-4) $(M(x, y, t) * M(y, z, s)) \leq M(x, z, t + s)$.
(FM-5) $M(x, y, \cdot) : (0, \infty) \to (0, 1]$ *is left continuous.*

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Gregori *et al.* [19] and Tirado *et al.* [20] introduced following fuzzy contraction.

Gregori *et al.* [19] Let $(X, M, *)$ be a fuzzy metric space. We say that the mapping $T : X \to X$ is fuzzy contractive if there exists $k \in (0, 1)$ such that

$$\frac{1}{M(Tx, Ty, t)} - 1 \leq k(\frac{1}{M(x, y, t)} - 1)$$

for each $x, y \in X$ and $t > 0$ (k is called the contractive constant of T).

Tirado *et al.* [20] Let $(K, M, *)$ be a fuzzy metric space. A mapping $S : K \to K$ is called a Tirado contraction, if the following inequality holds:

$$1 - M(Tx, Ty, t) \leq k(1 - M(x, y, t))$$

for each for each $x, y \in X$ and $t > 0$ and $k \in (0, 1)$.

Segre [3] defined the bicomplex number as $\xi = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 = z_1 + i_2 z_2$, where $a_1, a_2, a_3, a_4 \in \mathbb{C}_0$ (the set of reals) and $z_1 = a_1 + a_2 i_1, z_2 = a_3 + a_4 i_1 \in \mathbb{C}_1$ (the set of complex numbers), the independent units i_1, i_2 are such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$. We denote the set of bicomplex numbers as $\mathbb{C}_2$.

Pal *et al.* [21] defined the partial order relation $\preceq_{i_2}$ on $\mathbb{C}_2$ defined as:
For any $\xi = z_1 + i_2 z_2, \eta = w_1 + i_2 w_2 \in \mathbb{C}_2,\ \xi \preceq_{i_2} \eta$ if and only if $z_1 \preceq w_1$ and $z_2 \preceq w_2$ and $\xi \preceq_{i_2} \eta$ if one of the following conditions is satisfied:

(i) $z_1 = w_1, z_2 = w_2$,
(ii) $z_1 \prec w_1, z_2 = w_2$,
(iii) $z_1 = w_1, z_2 \prec w_2$,
(iv) $z_1 \prec w_1, z_2 \prec w_2$.

He also defined two conditions

1) Write $\xi \npreceq_{i_2} \eta$ if $\xi \preceq_{i_2} \eta$ and $\xi \neq \eta$ then one of $(ii), (iii)$ and (iv) is satisfied.
2) Write $\xi \prec_{i_2} \eta$ if only (iv) is satisfied.

Choi *et al.* [22] defined the bicomplex valued metric space as

Definition II.2. *[22] Let X is a nonempty set. Defined the mapping $d : X \times X \to \mathbb{C}_2$ satisfies the following conditions:*

1) $0 \prec_{i_2} d(x, y)$ *for all* $x, y \in X$,
2) $d(x, y) = 0$ *if and only if* $x = y$,
3) $d(x, y) = d(y, x)$*for all* $x, y \in X$,
4) $d(x, y) \preceq_{i_2} d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Then (X, d) is called a bicomplex valued metric spaces.

Singh *et al.* [2] defined the complex valued continuous t-norm and complex valued fuzzy metric space as

Definition II.3. *[2] A binary operation $* : r_s e^{i\theta} \times r_s e^{i\theta} \to r_s e^{i\theta}$, where in $r_s \in [0, 1]$ and a fix $\theta \in [0, \frac{\pi}{2}]$, is called complex valued continuous t-norm if it satisfies the following conditions:*

1) $*$ *is associative and commutative,*
2) $*$ *is continuous,*
3) $a * e^{i\theta} = a$, *for all* $a \in e^{i\theta}$, *where* $r_s \in [0, 1]$,
4) $a * b \preceq c * d$ *whenever* $a \preceq c$ *and* $b \preceq d$, *for all* $a, b, c, d \in r_s e^{i\theta}$, *where* $r_s \in [0, 1]$.

Example II.4. *[2]* $a * b = min(a, b)$.

Example II.5. *[2]* $a * b = max(a + b - e^{i\theta}, 0)$, *for a fix* $\theta \in [0, \frac{\pi}{2}]$.

Definition II.6. *[2] The triplet $(X, M, *)$ is said to be complex valued fuzzy metric space if X is an arbitrary non empty set, $*$ is a complex valued continuous t norm and $M : X \times X \times (0, \infty) \to r_s e^{i\theta}$ is a complex valued fuzzy set, where $r_s \in [0, 1]$ and $\theta \in [0, \frac{\pi}{2}]$ satisfying the following conditions:*
(BCF-1) $M(x, y, t) \succ 0$,
(BCF-2) $M(x, y, t) = e^{i\theta}$ *for all* $t > 0$ *iff* $x = y$,
(BCF-3) $M(x, y, t) = M(y, x, t)$,
(BCF-4) $M(x, y, t) * M(y, z, s) \succeq M(x, z, t + s)$,
(BCF-5) $M(x, y, \cdot) : (0, \infty) \to r_s e^{i\theta}$ *is continuous,*
for all $x, y, z \in X, s, t > 0, r_s \in [0, 1]$ *and* $\theta \in [0, \frac{\pi}{2}]$.
$(X, M, *)$ is called a complex valued fuzzy metric spaces.

Demir I. [1] presented the concept of a complex valued fuzzy b-metric space, which was inspired by Shukla *et al.* [23].

Definition II.7. *[1] Let X be a nonempty set, $s \geq 1$ a given real number, $*$ a continuous complex valued t-norm and M a complex fuzzy set on $X^2 \times P_\theta$ satisfying the following conditions:*
(bM1) $\theta \prec M(x,y,c)$,
(bM2) $M(x,y,c) = l$ *for every* $c \in P_\theta$ *if and only if* $x = y$,
(bM3) $M(x,y,c) = M(y,x,c)$,
(bM4) $M(x,y,c) * M(y,z,c') \preceq M(x,z,s(c+c'))$,
(bM5) $M(x,y,.) : P_\theta \to I$ *is continuous,*
for all $x,y,z \in X$ *and* $c,c' \in P_\theta$.
Then, the quadruple $(X,M,*,s)$ is called a complex valued fuzzy b-metric space and M is called a complex valued fuzzy b-metric on X.

The objective of this work is to define the notion of bicomplex valued fuzzy metric spaces and bicomplex valued fuzzy b-metric spaces with the help of continuous t-norms and prove a some fixed point theorems, we extend and improve some existing many recent fixed point theorems in the literature ([2], [3], [9], [22], [23]). We furnish examples to validate our result. Application is also provided to show the utility of our result.

III. Bicomplex valued fuzzy b-metric spaces

In this section, we introduce some new definitions. First of all we define bicomplex valued continuous t-norm and bicomplex valued fuzzy metric spaces, bicomplex valued fuzzy b-metric spaces and present some examples which certify our definitions.

Definition III.1. *A binary operation $* : r_s(1+i_2)e^{i_1\theta} \times r_s(1+i_2)e^{i_1\theta} \to r_s(1+i_2)e^{i_1\theta}$, where in $r_s \in [0,1]$ and a fix $\theta \in [\theta, \frac{\pi}{2}]$ is called bicomplex valued continuous t-norm if it satisfies the following conditions:*

1. $*$ *is associative and commutative,*
2. $*$ *be continuous,*
3. $a * (1+i_2)e^{i_1\theta} = a$, *for all* $a \in (1+i_2)e^{i_1\theta}$,
4. $a * b \preceq_{i_2} c * d$ *whenever* $a \preceq c$ *and* $b \preceq d$, *for all* $a,b,c,d \in r_s(1+i_2)e^{i_1\theta}$.

Definition III.2. *The triplet $(X,M,*)$ is said to be bicomplex valued fuzzy metric space if X is an arbitrary non empty set, $*$ is a bicomplex valued continuous t norm and $M : X \times X \times (0,\infty) \to r_s(1+i_2)e^{i_1\theta}$ is a bicomplex valued fuzzy set, satisfying the following conditions:*
(BVCF-1) $M(x,y,t) \succ_{i_2} 0$,
(BVCF-2) $M(x,y,t) = (1+i_2)e^{i_1\theta}$ *for all* $t > 0$ *iff* $x = y$,
(BVCF-3) $M(x,y,t) = M(y,x,t)$,
(BVCF-4) $M(x,y,t) * M(y,z,s) \succeq_{i_2} M(x,z,t+s)$,
(BVCF-5) $M(x,y,\cdot) : (0,\infty) \to r_s(1+i_2)e^{i_1\theta}$ *be continuous,*
for all $x,y,z \in X, s,t > 0, r_s \in [0,1]$ *and* $\theta \in [0, \frac{\pi}{2}]$.
$(X,M,*)$ *is called a bicomplex valued fuzzy metric spaces.*

Remark III.3. *If we take $(1+i_2)e^{i_1\theta} = e^{i\theta}$ then bicomplex valued fuzzy metric spaces goes to complex valued fuzzy metric spaces [2].*

Example III.4. *Let $X = \mathbb{R}$. We define $a * b = min\{a,b\}$, for all $a,b \in r_s(1+i_2)e^{i_1\theta}$, where $r_s \in [0,1]$ and $\theta \in [0, \frac{\pi}{2}]$ and $g : \mathbb{R}^+ \to (0,\infty)$ be an increasing continuous function.*

$$M(x,y,t) = (1+i_2)e^{i_1\theta} - \frac{|x-y|}{g(t)},$$

for all $x,y \in X$ and $t \in (0,\infty)$. Then $(X,M,*)$ is a bicomplex valued fuzzy metric spaces.

Example III.5. *Let $X = \mathbb{N}$. We define $a * b = max\{a+b-(1+i_2)e^{i_1\theta}, 0\}$, for all $a,b \in r_s(1+i_2)e^{i_1\theta}$, where $r_s \in [0,1]$ and $\theta \in [0, \frac{\pi}{2}]$ and*

$$M(x,y,t) = \begin{cases} (1+i_2)e^{i_1\theta} & \text{if } x = y, \\ txy(1+i_2)e^{i_1\theta} & \text{if } x \neq y \text{ and } t \leq 1, \\ xy(1+i_2)e^{i_1\theta} & \text{if } x \neq y \text{ and } t > 1, \end{cases}$$

for all $x,y \in X$ and $t \in (0,\infty)$. Then $(X,M,*)$ is a bicomplex valued fuzzy metric spaces.

The conception of a bicomplex valued fuzzy b-metric space as follows, inspired by the work of I. Demir [1].

Definition III.6. *The quadruple $(X,M,*,s)$ is said to be bicomplex valued fuzzy b-metric space if X is an arbitrary non empty set, $s \geq 1$ a given real number, $*$ is a bicomplex valued continuous t norm and $M : X \times X \times (0,\infty) \to r_s(1+i_2)e^{i_1\theta}$ is a bicomplex valued fuzzy set, satisfying the following conditions:*

(BVCFb-1) $M(x,y,c) \succ_{i_2} 0$,
(BVCFb-2) $M(x,y,c) = (1+i_2)e^{i_1\theta}$ *for all* $t>0$ *iff* $x=y$,
(BVCFb-3) $M(x,y,c) = M(y,x,c)$,
(BVCFb-4) $M(x,y,c) * M(y,z,c') \succeq_{i_2} M(x,z,s(c+c'))$,
(BVCFb-5) $M(x,y,\cdot) : (0,\infty) \to r_s(1+i_2)e^{i_1\theta}$ *is continuous,*
for all $x,y,z \in X, s,t>0, r_s \in [0,1]$ *and* $\theta \in [0,\frac{\pi}{2}]$.
$(X,M,*,s)$ *is called a bicomplex valued fuzzy b-metric spaces.*

Remark III.7. *If we take* $s=1$ *then bicomplex valued fuzzy b-metric space goes to bicomplex valued fuzzy metric spaces.*

Example III.8. *Let* $X=\mathbb{N}$. *We define* $a*b = max\{a+b-(1+i_2)e^{i_1\theta}, 0\}$, *for all* $a,b \in r_s(1+i_2)e^{i_1\theta}$, *where* $r_s \in [0,1]$ *and* $\theta \in [0,\frac{\pi}{2}]$ *and*

$$M(x,y,t) = \begin{cases} (1+i_2)e^{i_1\theta} & \textit{if } x=y, \\ (txy)^a(1+i_2)e^{i_1\theta} & \textit{if } x\neq y \textit{ and } t\leq 1, \\ (xy)^a(1+i_2)e^{i_1\theta} & \textit{if } x\neq y \textit{ and } t>1, \end{cases}$$

for all $x,y \in X, \quad a>0$ *and* $t \in (0,\infty)$. *Then* $(X,M,*,s)$ *is a bicomplex valued fuzzy b-metric spaces.*

Example III.9. *Let* $X=\mathbb{N}$. *We define* $a*b = max\{a+b-(1+i_2)e^{i_1\theta}, 0\}$, *for all* $a,b \in r_s(1+i_2)e^{i_1\theta}$, *where* $r_s \in [0,1]$ *and* $\theta \in [0,\frac{\pi}{2}]$ *and*

$$M(x,y,t) = \begin{cases} (1+i_2)e^{i_1\theta} & \textit{if } x=y, \\ (\frac{1}{(x-y)^2})^a \varphi(t) & \textit{if } x\neq y, \end{cases}$$

$\varphi : \mathbb{R}^+ \to (0,1]$ *be an increasing continuous function.*

$$\varphi(t) = \begin{cases} t & \textit{if } 0<t<1, \\ 1 & \textit{if } t\geq 1, \end{cases}$$

for all $x,y \in X, \quad a>0$ *and* $t \in (0,\infty)$. *Then* $(X,M,*,s)$ *is a bicomplex valued fuzzy b-metric spaces.*

Example III.10. *Let* $X=\mathbb{N}$ *and* $a*b = max\{a+b-(1+i_2)e^{i_1\theta}, 0\}$ *for all* $a,b \in r_s(1+i_2)e^{i_1\theta}$, *where* $r_s \in [0,1]$ *and* $\theta \in [0,\frac{\pi}{2}]$ *and*

$$M(x,y,t) = \begin{cases} (1+i_2)e^{i_1\theta} & \textit{if } x=y, \\ e^{\frac{-2^{n-1}t}{xy3^s}} & \textit{if } x\neq y, \quad x,y\in\mathbb{N}, \end{cases}$$

for all $x,y \in X, \quad s>0$ *and* $t \in (0,\infty)$. *Then* $(X,M,*,s)$ *is a bicomplex valued fuzzy b-metric spaces.*

Definition III.11. *Let* $(X,M,*,s)$ *be a bicomplex valued fuzzy b-metric spaces. We define an open ball* $B(x,r,c)$ *with centre* $x \in X$ *and radius* $r \in \mathbb{C}_2, 0 \prec r \prec (1+i_2)e^{i_1\theta}, c>0$ *as*

$$B(x,r,c) = \{y \in X : M(x,y,c) \succ (1+i_2)e^{i_1\theta}\},$$

where $\theta \in [0,\frac{\pi}{2}]$.

A point $x \in X$ *is called an interior point of set* $A \subset X$, *whenever there exists* $r \in \mathbb{C}_2, 0 \prec r \prec (1+i_2)e^{i_1\theta}$ *such that*

$$B(x,r,c) = \{y \in X : M(x,y,c) \succ (1+i_2)e^{i_1\theta} - r\} \subset A,$$

where $\theta \in [0,\frac{\pi}{2}]$.

The subset A *of* X *is called open whenever each element of* A *is an interior point of* A.

Definition III.12. *Let* $(X,M,*,s)$ *be a bicomplex valued fuzzy b-metric spaces. Then,* $(X,M,*,s)$ *is called a Hausdorff space if for any two distinct points* $p,q \in X$, *there exist two open balls* $B(p,r_1,c_1) = B(p,(1+i_2)e^{i_1\theta}-r_1,\frac{c}{2})$ *and* $B(q,r_2,c_2) = B(q,(1+i_2)e^{i_1\theta}-r_2,\frac{c}{2})$ *such that* $B(p,r_1,c_1) \cap B(q,r_2,c_2) = \phi$.

Definition III.13. *A sequence* $\{x_n\}$ *in a bicomplex valued fuzzy b-metric space* $(X,M,*,s)$ *is a Cauchy sequence if and only if*

$$\lim_{n\to\infty} M(x_{n+p},x_n,c) = (1+i_2)e^{i_1\theta}, p>0, c>0$$

or

$$\lim_{n\to\infty} |M(x_{n+p},x_n,c)| = 1, p>0, c>0.$$

Definition III.14. *A bicomplex valued fuzzy b-metric space in which every Cauchy sequence is convergent, is called bicomplex valued complete fuzzy b-metric spaces.*

Definition III.15. *Let $(X, M, *, s)$ be a bicomplex valued fuzzy b-metric space. We define a closed ball $B[x, r, c]$ with centre $x \in X$ and radius $r \in \mathbb{C}_2(0 \in r \in (1+i_2)e^{i_1\theta})$ and for all $c > 0$ by*

$$B[x, r, c] = \{y \in X : M(x, y, c) \in (1+i_2)e^{i_1\theta} - r\}.$$

Definition III.16. *A bicomplex valued fuzzy b-metric space in which, a sequence $\{x_n\} \in X$ is convergent to $x_n \to x$ if and only if $M(x_n, x, c) \to (1+i_2)e^{i_1\theta}$ as $n \to \infty$ or $|M(x_n, x, c)| \to 1$.*

Proposition III.17. *Every bicomplex valued fuzzy b-metric space is Hausdorff .*

Proof. Let $(X, M, *, s)$ be a bicomplex valued fuzzy b-metric space. Let p, q be two distinct points of X. Then

$$0 \prec M(x, y, c) \preceq (1+i_2)e^{i_1\theta}.$$

Let $M(x, y, t) = r$, for some $r \in \mathbb{C}_2$ then $0 \prec r_1 \prec (1+i_2)e^{i_1\theta}$. For each $r_0(r \prec r_0 \prec (1+i_2)e^{i_1\theta})$, we can find a $r_2(r_1 \prec (1+i_2)e^{i_1\theta})$ such that $r_2 * r_2 \succeq r_0$.

Now consider two open balls $B(p, (1+i_2)e^{i_1\theta} - r_1, \frac{c}{2s})$ and $B(q, (1+i_2)e^{i_1\theta} - r_2, \frac{c}{2s})$. Certainly

$$B(p, (1+i_2)e^{i_1\theta} - r_1, \frac{c}{2s}) \cap B(q, (1+i_2)e^{i_1\theta} - r_2, \frac{c}{2s}) = \phi.$$

If not then there exists

$$s \in B(p, (1+i_2)e^{i_1\theta} - r_1, \frac{c}{2s}) \cap B(q, (1+i_2)e^{i_1\theta} - r_2, \frac{c}{2s}).$$

Now consider

$$\begin{aligned} r &= M(x, y, c) \\ &\succeq M(p, z, \frac{c}{2s}) * M(z, q, \frac{c}{2s}) \\ &\succ r_1 * r_2 \quad \succeq r_0. \end{aligned}$$

Which is a contradiction. Therefore $(X, M, *, s)$ is Hausdorff. □

IV. Main Result

Now, we introduce bicomplex valued complete fuzzy b-metric spaces and extend the Banach fixed point theorem in the introduced space, as follows:

Definition IV.1. *Let $(X, M, *, s)$ be a bicomplex valued fuzzy b-metric space. We say that the mapping $T : X \to X$ is bicomplex fuzzy contractive if there exists $k \in (0, 1)$ such that*

$$\frac{1}{M(Tx, Ty, \frac{cs}{\lambda})} - \frac{1}{(1+i_2)e^{i_1\theta}} \preceq_{i_2} k\left(\frac{1}{M(x, y, c)} - \frac{1}{(1+i_2)e^{i_1\theta}}\right), \tag{1}$$

for each $x, y \in X$, $\quad 0 < \lambda < 1$ and $t > 0$ (k is called the bicomplex contractive constant of T).

Theorem IV.2. *Let $(X, M, *, s)$ be a bicomplex valued complete fuzzy b-metric space, for every sequence $\{c_n\}$ in $(1+i_2)e^{i_1\theta}$ such that*

$$\lim_{t\to\infty} M(x, y, c_n) = (1+i_2)e^{i_1\theta}, \quad \text{for all} \quad x, y \in X \text{ and } t > 0. \tag{2}$$

Let $T : X \to X$ be a mapping satisfying bicomplex fuzzy contractive. Then T has a unique fixed point.

Proof. Suppose T satisfies condition (1). Let a_0 be an arbitrary point in X and We define a sequence $\{a_n\}$ in X by

$$a_{n+1} = T_{a_n}, n = 0, 1, 2, \ldots$$

Applying condition (1) with $x = a_n$ and $y = a_{n+1}$, we have

$$\begin{aligned} \frac{1}{M(a_n, a_{n+1}, \frac{cs}{\lambda})} - \frac{1}{(1+i_2)e^{i_1\theta}} &= \frac{1}{M(Ta_{n-1}, a_n, c)} - \frac{1}{(1+i_2)e^{i_1\theta}} \\ &\preceq_{i_2} k\left(\frac{1}{M(a_{n-1}, a_n, \frac{cs}{\lambda})} - \frac{1}{(1+i_2)e^{i_1\theta}}\right) \\ &\preceq_{i_2} k^2\left(\frac{1}{M(a_{n-2}, a_{n-1}, \frac{cs^2}{\lambda^2})} - \frac{1}{(1+i_2)e^{i_1\theta}}\right) \\ &\quad\vdots \\ &\preceq_{i_2} k^n\left(\frac{1}{M(a_0, a_1, \frac{cs^{n+1}}{\lambda^{n+1}})} - \frac{1}{(1+i_2)e^{i_1\theta}}\right). \end{aligned}$$

Thus for any positive integer m and using $(BVCFb-4)$, we have

$$\begin{aligned}\frac{1}{M(a_n,a_{n+m},c)}-\frac{1}{(1+i_2)e^{i_1\theta}} \preceq_{i_2} & \ k(\frac{1}{M(a_n,a_{n+1},\frac{cs}{\lambda})}-\frac{1}{(1+i_2)e^{i_1\theta}}* \\ & \cdots*(\frac{1}{M(a_{n+m-1},a_{n+m},t)}-\frac{1}{(1+i_2)e^{i_1\theta}})) \\ \preceq_{i_2} & \ k^n(\frac{1}{M(a_0,a_1,t)}-\frac{1}{(1+i_2)e^{i_1\theta}})* \\ & \cdots*(\frac{1}{M(a_0,a_1,\frac{cs^{n+1}}{\lambda^{n+1}})}-\frac{1}{(1+i_2)e^{i_1\theta}}))). \end{aligned} \tag{3}$$

Which on letting $n\to\infty$, reduces to

$$\lim_{n\to\infty}(\frac{1}{M(a_0,a_1,\frac{cs^{n+1}}{\lambda^{n+1}})}-(1+i_2)e^{i_1\theta})\preceq_{i_2}0,$$

Equation (3) gives rise to

$$\begin{aligned}\lim_{n\to\infty}(\frac{1}{M(a_n,a_{n+m},c)}-\frac{1}{(1+i_2)e^{i_1\theta}})\preceq_{i_2} & \ k(\frac{1}{(1+i_2)e^{i_1\theta}}-\frac{1}{(1+i_2)e^{i_1\theta}})* \\ & \cdots*(\frac{1}{(1+i_2)e^{i_1\theta}}-\frac{1}{(1+i_2)e^{i_1\theta}}),\end{aligned} \tag{4}$$

which implies that

$$\lim_{n\to\infty}(\frac{1}{M(a_n,a_{n+m},c)}-(1+i_2)e^{i_1\theta})\preceq_{i_2}0.$$

In view of Definition III.13, we conclude that

$$\lim_{n\to\infty}M(a_n,a_{n+m},c)=(1+i_2)e^{i_1\theta}.$$

We assert that $\{a_n\}$ be Cauchy sequence in X. Since X is complete, then essentially $a_n\to u$ as $n\to\infty$, where $u\in X$. Consequently

$$\begin{aligned}\frac{1}{M(Tu,u,c)}-\frac{1}{(1+i_2)e^{i_1\theta}}\preceq_{i_2} & \ k(\frac{1}{M(Tu,a_{n+1},\frac{c}{2s})}-\frac{1}{(1+i_2)e^{i_1\theta}}) \\ & *(\frac{1}{M(a_{n+1},u,\frac{c}{2s})}-\frac{1}{(1+i_2)e^{i_1\theta}})) \\ \preceq_{i_2} & \ k(\frac{1}{M(Tu,Ta_n,\frac{c}{2s})}-\frac{1}{(1+i_2)e^{i_1\theta}}) \\ & *(\frac{1}{M(a_{n+1},u,\frac{c}{2s})}-\frac{1}{(1+i_2)e^{i_1\theta}}) \\ \preceq_{i_2} & \ k(\frac{1}{M(u,a_n,\frac{c}{2s})}-\frac{1}{(1+i_2)e^{i_1\theta}}) \\ & *(\frac{1}{M(a_{n+1},u,\frac{c}{2s})}-\frac{1}{(1+i_2)e^{i_1\theta}}).\end{aligned}$$

Letting $n\to\infty$, we have

$$\frac{1}{M(Tu,u,c)}-\frac{1}{(1+i_2)e^{i_1\theta}}\preceq_{i_2}(\frac{1}{M(u,u,\frac{cs}{\lambda})}-\frac{1}{(1+i_2)e^{i_1\theta}})*(\frac{1}{M(u,u,\frac{cs^n}{\lambda^n})}-\frac{1}{(1+i_2)e^{i_1\theta}}).$$

Now by Definition III.6 $(BVCFb-2)$, we have

$$\frac{1}{M(Tu,u,c)}-\frac{1}{(1+i_2)e^{i_1\theta}}\preceq_{i_2}(\frac{1}{(1+i_2)e^{i_1\theta}}-\frac{1}{(1+i_2)e^{i_1\theta}})*(\frac{1}{(1+i_2)e^{i_1\theta}}-\frac{1}{(1+i_2)e^{i_1\theta}}).$$

Or

$$M(Tu,u,c)=(1+i_2)e^{i_1\theta}.$$

Which implies that $Tu = u$. Thus u is a fixed point of T . According to the uniqueness of fixed point, assume $w \in X$ be another fixed point of T such that $w \neq u$. The inequality turns into

$$\begin{aligned}\frac{1}{(1+i_2)e^{i_1\theta}} &\preceq_{i_2} \frac{1}{M(u,w,c)} - \frac{1}{(1+i_2)e^{i_1\theta}}\\ &= \frac{1}{M(Tu,Tw,c)} - \frac{1}{(1+i_2)e^{i_1\theta}}\\ &\preceq_{i_2} k(\frac{1}{M(u,w,\frac{cs}{\lambda})} - \frac{1}{(1+i_2)e^{i_1\theta}})\\ &\preceq_{i_2} k(\frac{1}{M(u,w,\frac{cs^2}{\lambda^2}))} - \frac{1}{(1+i_2)e^{i_1\theta}})\\ &\vdots\\ &\preceq_{i_2} k(\frac{1}{M(u,w,\frac{cs^n}{\lambda^n}))} - \frac{1}{(1+i_2)e^{i_1\theta}}),\end{aligned}$$

which implies that

$$\frac{1}{(1+i_2)e^{i_1\theta}} \preceq_{i_2} \frac{1}{M(u,w,c)} - \frac{1}{(1+i_2)e^{i_1\theta}}.$$

Thus we obtain

$$(1+i_2)e^{i_1\theta} \succeq_{i_2} M(u,w,c).$$

Since $k < 1$, then on making $n \to \infty$, we gets $u = w$. Thus, we conclude that T has a unique fixed point. □

Remark IV.3. *For* $\frac{1}{(1+i_2)e^{i_1\theta}} = 1, \quad \frac{cs}{\lambda} = c$ *and* $s = 1$, *we get the following corollary.*

Corollary IV.4. *[19] Let* $(X, M, *)$ *be a fuzzy metric space. We say that the mapping* $T : X \to X$ *is fuzzy contractive if there exists* $k \in (0, 1)$ *such that*

$$\frac{1}{M(Tx,Ty,t)} - 1 \leq k(\frac{1}{M(x,y,t)} - 1)$$

for each $x, y \in X$ *and* $t > 0$ *(*k *is called the contractive constant of* T *).*

Now, we furnish an example which shows the superiority of our result.

Example IV.5. *Let* $X = \mathbb{N}$ *and* $a * b = max\{a + b - (1 + i_2)e^{i_1\theta}, 0\}$ *for all* $a, b \in r_s(1 + i_2)e^{i_1\theta}$, *where* $r_s \in [0, 1]$ *and* $\theta \in [0, \frac{\pi}{2}]$ *and*

$$M(x,y,t) = \begin{cases}(1+i_2)e^{i_1\theta} & \text{if } x = y,\\ e^{\frac{-2^{n-1}t}{xy3^s}} & \text{if } x \neq y, \quad x,y \in \mathbb{N},\end{cases}$$

for all $x, y \in X, \quad s > 0$ *and* $t \in (0, \infty)$. *Then* $(X, M, *, s)$ *is a bicomplex valued fuzzy b-metric spaces. Certainly here*

$$\lim_{c\to\infty} M(x,y,c) = (1+i_2)e^{i_1\theta},$$

for all $x, y \in X$ *and* $t \in (0, \infty)$. *Define* $T(x) = 4x$. *By a routine calculation, one can verify that* T *satisfies the condition*

$$\frac{1}{M(Tx,Ty,\frac{cs}{\lambda})} - \frac{1}{(1+i_2)e^{i_1\theta}} \preceq_{i_2} \frac{1}{M(x,y,c)} - \frac{1}{(1+i_2)e^{i_1\theta}}. \tag{5}$$

Thus all the conditions of Theorem IV.2 are satisfied and $x = 0$ *is the unique fixed point of* T.

V. A (ψ, ϕ) Contraction Function

Now, we introduce modified (ϕ, ψ) fuzzy contraction in bicomplex valued complete fuzzy b-metric spaces, as follows:

Definition V.1. *Let* $(X, M, *, s)$ *be a bicomplex valued fuzzy b-metric space. Let* $T : X \to X$ *be satisfies the following*

$$\psi(\frac{1}{M(Tx,Ty,c)} - \frac{1}{(1+i_2)e^{i_1\theta}}) \preceq_{i_2} \psi(\frac{1}{M(x,y,c)} - \frac{1}{(1+i_2)e^{i_1\theta}}) \tag{6}$$

$$- \phi(\frac{1}{M(x,y,c)} - \frac{1}{(1+i_2)e^{i_1\theta}}), \tag{7}$$

for all $x, y \in X$ *and* $0 < k < 1$, *such that*

(i) ψ *is continuous and non-decreasing with* $\psi(t) = 0$ *if and only if* $t = 0$.

(ii) *ϕ is continuous and decreasing with $\phi(t) = 0$ if and only if $t = 0$. f is called $(\psi - \phi)-$ contraction function.*

Theorem V.2. *Let $(X, M, *, s)$ be a bicomplex valued fuzzy b-metric spaces. Let $T : X \to X$ be satisfies the following*

$$\psi(\frac{1}{M(Tx,Ty,c)} - \frac{1}{(1+i_2)e^{i_1\theta}}) \preceq_{i_2} \psi(\frac{1}{M(x,y,c)} - \frac{1}{(1+i_2)e^{i_1\theta}}) - \phi(\frac{1}{M(x,y,c)} - \frac{1}{(1+i_2)e^{i_1\theta}}). \tag{8}$$

Then T has a unique fixed point.

Proof. Suppose T satisfies condition (6). Let a_0 be an arbitrary point in X and We define a sequence $\{a_n\}$ in X. Applying condition (6) with $x = a_{n-1}$ and $y = a_n$, we have

$$\psi(\frac{1}{M(Ta_{n-1},Ta_n,c)} - \frac{1}{(1+i_2)e^{i_1\theta}}) \preceq_{i_2} \psi(\frac{1}{M(a_{n-1},a_n,c)} - \frac{1}{(1+i_2)e^{i_1\theta}}) - \phi(\frac{1}{M(a_{n-1},a_n,c)} - \frac{1}{(1+i_2)e^{i_1\theta}}),$$

which implies that

$$\psi(\frac{1}{M(a_n,a_{n+1},c)} - \frac{1}{(1+i_2)e^{i_1\theta}}) \preceq_{i_2} \psi(\frac{1}{M(a_{n-1},a_n,c)} - \frac{1}{(1+i_2)e^{i_1\theta}}) - \phi(\frac{1}{M(a_{n-1},a_n,c)} - \frac{1}{(1+i_2)e^{i_1\theta}}).$$

If $M(a_{n-1}, a_n, c) = (1+i_2)e^{i_1\theta}$, Otherwise, if $M(a_{n-1}, a_n, c) < (1+i_2)e^{i_1\theta}$, then

$$\psi(\frac{1}{M(a_n,a_{n+1},c)} - \frac{1}{(1+i_2)e^{i_1\theta}}) \prec_{i_2} \psi(\frac{1}{M(a_{n-1},a_n,t)} - \frac{1}{(1+i_2)e^{i_1\theta}}),$$

since ψ is decreasing. Thus for any positive integer m and using definition III.6 $(BVCFb-4)$, we have

$$\begin{aligned} M(a_n, a_{n+m}, c) &\succeq_{i_2} M(a_n, a_{n+1}, \frac{cs}{\lambda}) * M(a_{n+1}, a_{n+2}, \frac{cs}{\lambda}) * \\ &\cdots * M(a_{n+p-1}, a_{n+p}, \frac{cs}{\lambda}) \\ &\succeq_{i_2} (1+i_2)e^{i_1\theta} * \cdots * (1+i_2)e^{i_1\theta}. \end{aligned}$$

We conclude that

$$\lim_{n\to\infty} M(a_n, a_{n+m}, c) = (1+i_2)e^{i_1\theta}.$$

We assert that $\{a_n\}$ is Cauchy sequence in X. Since X is complete, then essentially $a_n \to u$ as $n \to \infty$, where $u \in X$.

$$\psi(\frac{1}{M(Tu,u,c)} - \frac{1}{(1+i_2)e^{i_1\theta}}) \preceq_{i_3} \psi(\frac{1}{M(u,u,c)} - \frac{1}{(1+i_2)e^{i_1\theta}}) - \phi(\frac{1}{M(u,u,c)} - \frac{1}{(1+i_2)e^{i_1\theta}}).$$

If $M(Tu, u, c) = (1+i_2)e^{i_1\theta}$, Otherwise, if $M(Tu, u, c) < (1+i_2)e^{i_1\theta}$, then

$$\psi(\frac{1}{M(Tu,u,c)} - \frac{1}{(1+i_2)e^{i_1\theta}}) \prec_{i_2} \psi(\frac{1}{M(u,u,c)} - \frac{1}{(1+i_2)e^{i_1\theta}}),$$

Or

$$M(Tu, u, c) = (1+i_2)e^{i_1\theta}.$$

Which implies that $Tu = u$. Thus u is a fixed point of T . According to the uniqueness of fixed point, assume $w \in X$ be another fixed point of T such that $w \neq u$. The inequality turns into

$$\psi(\frac{1}{M(Tu,w,c)} - \frac{1}{(1+i_2)e^{i_1\theta}}) \preceq_{i_2} \psi(\frac{1}{M(u,w,c)} - \frac{1}{(1+i_2)e^{i_1\theta}}),$$

thus we obtain

$$(1+i_2)e^{i_1\theta} \succeq_{i_2} M(u, w, c).$$

Since $k < 1$, then on making $n \to \infty$, we gets $u = w$. Thus, we conclude that T has a unique fixed point. ☐

Remark V.3. *For* $\psi(t) = t, \quad \phi(t) = (1-k)t$ *and* $k \in (0,1), \quad t \in (0,\infty)$ *in Theorem V.2, then we get Theorem IV.2.*

We present example validates the aforesaid theorem.

Example V.4. *Let* $X = \mathbb{N}$. *We define* $a * b = max\{a + b - (1+i_2)e^{i_1\theta}, 0\}$, *for all* $a, b \in r_s(1+i_2)e^{i_1\theta}$, *where* $r_s \in [0,1]$ *and* $\theta \in [0, \frac{\pi}{2}]$ *and*

$$M(x,y,t) = \begin{cases} (1+i_2)e^{i_1\theta} & \text{if } x = y, \\ (txy)^a(1+i_2)e^{i_1\theta} & \text{if } x \neq y \text{ and } t \leq 1, \\ (xy)^a(1+i_2)e^{i_1\theta} & \text{if } x \neq y \text{ and } t > 1, \end{cases}$$

for all $x, y \in X, \quad a > 0, \quad \psi(t) = t, \quad \phi(t) = (1-k)t$ *and* $k \in (0,1), \quad t \in (0,\infty)$. *Then* $(X, M, *, s)$ *is a bicomplex valued fuzzy b-metric Then* $(X, M, *, s)$ *is a bicomplex valued fuzzy b-metric spaces.*

VI. Application

In this section, we give an application to support the efficacy of our established result finding the unique solution to a system of equation that appears in dynamic programming.

Let U and V be Banach spaces, $W \subseteq U$ be a state space and $D \subseteq V$ be a decision space. It is desirable to find an optimal decision in the given state space using dynamic programming, we refer to [24], [25]. Dynamic programming offers helpful resources for both computer programming and mathematics optimization. Particularly, the challenge of dynamic programming in multistage processes is reduced to the challenge of solving the functional equation.

$$\begin{cases} p(x) = \sup_{y \in D}\{g(x,y) + \phi(x,y,p(\xi(x,y)))\} & \text{for } x \in W, \\ q(x) = \sup_{y \in D}\{h(x,y) + \psi(x,y,q(\xi(x,y)))\} & \text{for } x \in W, \end{cases} \tag{9}$$

where $\xi : W \times D \to W, \quad g, \ h : W \times D \to \mathbb{R}$ and $\phi, \ \psi : W \times D \times \mathbb{R} \to \mathbb{R}$. We establish existence and uniqueness of a bounded solution of Equation (9). For $h, k \in B(W)$. Denote $B(W)$ the set of all bounded real valued function on W, and $\theta \in [0, \frac{\pi}{2}], \ r \in [0,1)$ define

$$M(x,y,t) = (1+i_2)e^{i_1\theta}e^{\frac{-d(x,y)}{rt}}, \tag{10}$$

where

$$d(x,y) = \sup_{x \in W}|h(x) - k(x)| = ||h - k||.$$

Then $(B(W), M, *, s)$ is complete bicomplex valued fuzzy b-metric spaces. For every $(x,y) \in W \times D, \quad h, k \in B(W)$ and $x \in W$, define

$$Th(x) = \sup_{y \in D}\{g(x,y) + \phi(x,y,h(\xi(x,y)))\} \quad for \ x \in W \tag{11}$$

and

$$Sh(x) = \sup_{y \in D}\{h(x,y) + \psi(x,y,h(\xi(x,y)))\} \quad for \ x \in W. \tag{12}$$

Suppose that the following condition hold:
(1) $\phi, \ \psi, \ g$ and h are bounded and continuous.
(2) There exist $r \in [0,1)$ and

$$\Big(\frac{1}{(1+i_2)e^{i_1\theta}}e^{\frac{d(\phi(x,y_2,h(\xi(x,y))) - \phi(x,y_1,k(\xi(x,y))))}{rt}} - \frac{1}{(1+i_2)e^{i_1\theta}}\Big) \preceq_{i_2} k\Big(\frac{1}{(1+i_2)e^{i_1\theta}}e^{\frac{d(Sh(x),Sk(x)))}{t}} - \frac{1}{(1+i_2)e^{i_1\theta}}\Big). \tag{13}$$

(3) For any $h \in B(W)$ and $x \in W$, there exists $k \in B(W)$ such that $Th(x) = Sk(x)$.
(4) There exists $h \in B(W)$ such that $Th(x) = Sk(x)$ implies that $STh(x) = TSk(x)$.

Theorem VI.1. *If condition* $(1) - (4)$ *are satisfied, then the system of functional equations* (9) *has a unique solution and bounded in* $B(W)$.

Proof. T is a self-map on $B(W)$ and $(B(W), M, *, s)$ is a complete bicomplex valued fuzzy b-metric spaces, then for every real number α and $x \in W$, there exist $y_1, y_2 \in D$ such that

$$T(h(x)) < g(x,y_1) + \phi(x,y_1,h(\xi_1)) + \alpha \tag{14}$$

and

$$T(k(x)) < g(x, y_2) + \phi(x, y_2, h(\xi_2)) + \alpha, \tag{15}$$

where $\xi_1 = \xi(x, y_1)$ and $\xi_2 = \xi(x, y_2)$. From Equations (14) and (15), we obtain

$$T(h(x)) \geq g(x, y_2) + \phi(x, y_1, h(\xi_1)) + \alpha \tag{16}$$

and

$$T(k(x)) \geq g(x, y_1) + \phi(x, y_2, k(\xi_2)) + \alpha. \tag{17}$$

Then, Equations (16) and (17) with Equation (13), implies that

$$\begin{aligned} T(k(x)) - T(h(x)) &< \phi(x, y_2, k(\xi_2)) - \phi(x, y_1, h(\xi_1)) + \alpha \\ &\leq |\phi(x, y_2, k(\xi_2)) - \phi(x, y_1, h(\xi_1))| + \alpha. \end{aligned}$$

Since $\alpha > 0$ as an arbitrary number, we obtain

$$\begin{aligned} d(T((h(x)), T(k(x))) &= |T((h(x)) - T(k(x)))| \\ &\leq |\phi(x, y_2, k(\xi_2)) - \phi(x, y_1, h(\xi_1))|. \end{aligned}$$

Thus

$$\begin{aligned} (\frac{1}{(1+i_2)e^{i_1\theta}} e^{\frac{d(T((h(x)),T(k(x)))}{rt}} - \frac{1}{(1+i_2)e^{i_1\theta}}) &\preceq_{i_2} (\frac{1}{(1+i_2)e^{i_1\theta}} e^{\frac{|\phi(x,y_2,k(\xi_2))-\phi(x,y_1,h(\xi_1))|}{rt}} \\ &\quad - \frac{1}{(1+i_2)e^{i_1\theta}}) \\ &\preceq_{i_2} (\frac{1}{(1+i_2)e^{i_1\theta}} e^{\frac{d(\phi(x,y_2,k(\xi_2)),\phi(x,y_1,h(\xi_1)))}{rt}} \\ &\quad - \frac{1}{(1+i_2)e^{i_1\theta}}) \\ &\preceq_{i_2} k(\frac{1}{(1+i_2)e^{i_1\theta}} e^{\frac{d(Sh(x),Sk(x)))}{t}} \\ &\quad - \frac{1}{(1+i_2)e^{i_1\theta}}). \end{aligned} \tag{18}$$

Then it follows that

$$\begin{aligned} (\frac{1}{(1+i_2)e^{i_1\theta} e^{\frac{-(d(T((h(x)),T(k(x))))}{rt}}} - \frac{1}{(1+i_2)e^{i_1\theta}}) &\preceq_{i_2} k(\frac{1}{(1+i_2)e^{i_1\theta} e^{\frac{-(d(Sh(x),Sk(x)))}{t}}} \\ &\quad - \frac{1}{(1+i_2)e^{i_1\theta}}). \end{aligned}$$

Therefore

$$\begin{aligned} (\frac{1}{M(T(h(x)), T(k(x)), rt)} - \frac{1}{(1+i_2)e^{i_1\theta}}) &\preceq_{i_2} k(\frac{1}{M(S(h(x)), S(k(x)), t)} \\ &\quad - \frac{1}{(1+i_2)e^{i_1\theta}}). \end{aligned}$$

Therefore all the conditions of Theorem IV.2 are satisfied. As a result, the mapping T has a unique fixed point $x \in X$. Hence the system of functional equations has a unique and bounded solution. □

VII. Conclusion

In this article, motivated and inspired by the work of I. Demir [1], Ramot *et al.* [18], Choi *et al.* [22] and Ismat Beg *et al.* [26], we introduce (ψ, ϕ) contraction in bicomplex valued fuzzy b-metric spaces. Our results extend and generalize some results in the literature by [1], [2]. Our investigations and results obtained were supported by suitable examples. We also provide an application of our established result to find the unique solution of the system of equation arises in dynamic programming. This work opens a new path for researchers in the concerned field.

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