Some Classes of Co-Schwarzian Functions and Its Coefficient Inequality that Is Sharp

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Abstract — In our present work, we defined an inequality called Fekete – Szegő Inequality for functions $f(z)$ in the classes of starlike functions and convex functions along with subclasses of these classes.

Keywords — Bounded analytic functions and concept of subordination, Convex functions, Fekete–Szegő Inequality, Starlike functions.

I. INTRODUCTION

We deal with geometric function theory and Koebe [6] proved that Riemann Mapping theorem is the main pillar of this theory. From this theorem, a conjecture called Bieberbach conjecture was produced. This conjecture was given by L. Bieberbach but finally proved by Louis De Branges[1]. While tackling with this conjecture, an equality arises called Fekete–Szegő Inequality, which was given by M. Fekete and G. Szegő [3]. Till now many researchers have solved this inequality for various classes and subclasses of starlike functions, convex functions, close-to-convex functions and for many other functions. Now, here we establish this inequality for different classes and subclasses of starlike functions and convex functions.

Firstly, we define some fundamental classes.

A consists of the family of analytic functions f with the normalization $f(0)=0, f'(0)=1$ and having functions of the type $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$;

S be the family of functions f normalized by $f(0)=0, f'(0)=1$ where $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$; univalent in the open disk: $E = \{z \in \mathbb{C} : |z| < 1\}$ and $S^*(\phi)$ and $C(\phi)$ be the class of functions $f \in S$ for which $\frac{zf'(z)}{f(z)} < \phi(z)$ and $\frac{[zf'(z)]'}{[f(z)]'} < \phi(z)$; this was introduced by Ma and Minda [10].

In this equation, “ $<$ ” denotes subordination [which states that let $f(z)$ and $g(z)$ are two analytic functions, if there exists a Schwarzian function $w(z)$ (analytic in E) in such a way that $|w(z)| < 1$, $w(0)=0$ and $f(z) = g(w(z))$; $z \in E$, then the function $f(z)$ is subordinate to $g(z)$ and we write it as $f(z) < g(z)$].

The concept of subordination was given by Lindelof [7]. Here, $\phi(z)$ is an analytic function with positive real part on E which maps the unit disk E onto a region starlike with respect to 1 as well as symmetric with respect to real axis, satisfying conditions $\phi(0)=0$ and $\phi'(0) > 0$ and Schwarzian function is an analytic function of the type $w(z) = \sum_{n=1}^{\infty} c_n z^n$ with conditions $w(0)=0$ and $|w(z)| < 1$.

Miller, S.S., Mocanu, P.T. and Reade, M.O. [9] proved the conditions $|c_1| \leq 1$, $|c_2| \leq 1/|c_1|^2$ for the above defined bounded analytic functions.

The class defined below is denoted by $TK[\alpha, \beta]$ and is a subclass of K , satisfying the condition along with some subclasses:

$$(1-\alpha)\frac{f(z)}{z} + \alpha \frac{\{zf'(z) + \beta z^2 f''(z)\}'}{\{f(z)\}'} < \phi(z).$$

II. MAIN RESULT

THEOREM-1:

Let $f(z) \in TK[\alpha, \beta]$ and $\phi(z) = \frac{1+w(z)}{1-w(z)}$; $w(z)$ is a Schwarzian function, then:

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$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{16\alpha(1+2\beta) + 2[1 + \alpha(1+4\beta)]^2}{[1 + \alpha(1+4\beta)]^2(1+5\alpha+18\alpha\beta)} - \frac{4\mu}{[1 + \alpha(1+4\beta)]^2}; \\ \mu \leq \frac{4\alpha(1+2\beta)}{1+5\alpha+18\alpha\beta}; \\ \frac{2}{1+5\alpha+18\alpha\beta}; \frac{4\alpha(1+2\beta)}{1+5\alpha+18\alpha\beta} \leq \mu \\ \leq \frac{4\alpha(1+2\beta) + [1 + \alpha(1+4\beta)]^2}{(1+5\alpha+18\alpha\beta)}; \\ \frac{4\mu}{[1 + \alpha(1+4\beta)]^2} - \frac{16\alpha(1+2\beta) + 2[1 + \alpha(1+4\beta)]^2}{[1 + \alpha(1+4\beta)]^2(1+5\alpha+18\alpha\beta)}; \\ \mu \geq \frac{4\alpha(1+2\beta) + [1 + \alpha(1+4\beta)]^2}{(1+5\alpha+18\alpha\beta)}. \end{cases}$$

The result is sharp.

PROOF:

By definition of TK $[\alpha, \beta]$,

$$(1-\alpha)\frac{f(z)}{z} + \alpha\frac{\{zf'(z) + \beta z^2 f''(z)\}'}{\{f(z)\}'} = \frac{1+w(z)}{1-w(z)} \quad (1)$$

where

$$w(z) = c_1 z + c_2 z^2 + c_3 z^3$$

$$f(z) = z + a_2 z^2 + a_3 z^3$$

$$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3$$

$$f''(z) = 2a_2 + 6a_3 z + 12a_4 z^2$$

Now by putting all these values in (1), we get:

$$(1-\alpha)(1+a_2 z + a_3 z^2 + \dots) + \alpha \left(\frac{\{z + 2a_2 z^2 + 3a_3 z^3 + \beta(2a_2 z + 6a_3 z + 12a_4 z^2)\}'}{\{z + a_2 z^2 + a_3 z^3\}'} \right) = \frac{1 + c_1 z + c_2 z^2 + \dots}{1 - (c_1 z + c_2 z^2 + \dots)}$$

By expanding the series, we get:

$$\begin{aligned} & 1 + [1 + \alpha(1+4\beta)]a_2 z + [(1 + 5\alpha + 18\alpha\beta)a_3 - 4\alpha(1 + 2\beta)a_2^2]z^2 \\ & = 1 + 2c_1 z + 2(c_2 + c_1^2)z^2 \end{aligned}$$

By comparing, we get:

$$\begin{aligned} a_2 &= \frac{2c_1}{1 + \alpha(1+4\beta)} \\ a_3 &= \frac{2[1 + \alpha(1+4\beta)]^2 c_2 + [16\alpha(1+2\beta) + 2\{1 + \alpha(1+4\beta)\}^2] c_1^2}{(1 + 5\alpha + 18\alpha\beta)\{1 + \alpha(1+4\beta)\}^2} \end{aligned}$$

Using these values of a_2 and a_3 , we get:

$$a_3 - \mu a_2^2 = \frac{2c_2}{1+5\alpha+18\alpha\beta} + \left(\frac{16\alpha(1+2\beta) + 2\{1 + \alpha(1+4\beta)\}^2}{(1+5\alpha+18\alpha\beta)\{1 + \alpha(1+4\beta)\}^2} - \frac{4\mu}{\{1 + \alpha(1+4\beta)\}^2} \right) c_1^2$$

After applying mode on both sides, we get:

$$|a_3 - \mu a_2^2| \leq \left(\frac{2}{1+5\alpha+18\alpha\beta} \right) |c_2| + \left| \frac{16\alpha(1+2\beta) + 2\{1+\alpha(1+4\beta)\}^2}{(1+5\alpha+18\alpha\beta)\{1+\alpha(1+4\beta)\}^2} - \frac{4\mu}{\{1+\alpha(1+4\beta)\}^2} \right| |c_1|^2$$

Using $|c_2| \leq 1 - |c_1|^2$, we get:

$$|a_3 - \mu a_2^2| \leq \frac{2}{1+5\alpha+18\alpha\beta} + \left\{ \left| \frac{16\alpha(1+2\beta) + 2\{1+\alpha(1+4\beta)\}^2}{(1+5\alpha+18\alpha\beta)\{1+\alpha(1+4\beta)\}^2} - \frac{4\mu}{\{1+\alpha(1+4\beta)\}^2} \right| - \frac{2}{1+5\alpha+18\alpha\beta} \right\} |c_1|^2$$

Case 1:

$$\mu \leq \frac{8\alpha(1+2\beta) + \{1+\alpha(1+4\beta)\}^2}{2(1+5\alpha+18\alpha\beta)}$$

Then

$$|a_3 - \mu a_2^2| \leq \frac{2}{1+5\alpha+18\alpha\beta} + \left| \frac{16\alpha(1+2\beta)}{(1+5\alpha+18\alpha\beta)\{1+\alpha(1+4\beta)\}^2} - \frac{4\mu}{\{1+\alpha(1+4\beta)\}^2} \right| |c_1|^2$$

Subcase – 1 (a):

$$\mu \leq \frac{4\alpha(1+2\beta)}{1+5\alpha+18\alpha\beta}$$

Using $|c_1| \leq 1$, we get:

$$|a_3 - \mu a_2^2| \leq \frac{2\{1+\alpha(1+4\beta)\}^2 + 16\alpha(1+2\beta)}{(1+5\alpha+18\alpha\beta)\{1+\alpha(1+4\beta)\}^2} - \frac{4\mu}{\{1+\alpha(1+4\beta)\}^2} \quad (2)$$

Subcase – 1 (b):

$$\mu \geq \frac{4\alpha(1+2\beta)}{1+5\alpha+18\alpha\beta}$$

Then

$$|a_3 - \mu a_2^2| \leq \frac{2}{1+5\alpha+18\alpha\beta} \quad (3)$$

Case – 2:

$$\mu \geq \frac{8\alpha(1+2\beta) + \{1+\alpha(1+4\beta)\}^2}{2(1+5\alpha+18\alpha\beta)}$$

Then

$$|a_3 - \mu a_2^2| \leq \frac{2}{1+5\alpha+18\alpha\beta} + \left| \frac{4\mu}{\{1+\alpha(1+4\beta)\}^2} - \frac{16\alpha(1+2\beta) + 4\{1+\alpha(1+4\beta)\}^2}{(1+5\alpha+18\alpha\beta)\{1+\alpha(1+4\beta)\}^2} \right| |c_1|^2$$

Subcase-2 (a):

$$\mu \geq \frac{4\alpha(1+2\beta) + \{1+\alpha(1+4\beta)\}^2}{(1+5\alpha+18\alpha\beta)}$$

$$\text{Then } |a_3 - \mu a_2^2| \leq -\frac{2\{1+\alpha(1+4\beta)\}^2 + 16\alpha(1+2\beta)}{(1+5\alpha+18\alpha\beta)\{1+\alpha(1+4\beta)\}^2} + \frac{4\mu}{\{1+\alpha(1+4\beta)\}^2} \quad (4)$$

Subcase – 2 (b):

$$\mu \leq \frac{4\alpha(1+2\beta) + \{1+\alpha(1+4\beta)\}^2}{(1+5\alpha+18\alpha\beta)}$$

Then

$$|a_3 - \mu a_2^2| \leq \frac{2}{1+5\alpha+18\alpha\beta} \quad (5)$$

Combining (2), (3), (4) and (5), we get the required result.

Extremal functions of this inequality are given by:

$$f(z) = z \left(1 - \frac{2\alpha[(\alpha+5) + 2\beta\{3 + 4\alpha(1+2\beta)\}]}{(1+\alpha+4\alpha\beta)(1+5\alpha+18\alpha\beta)} z \right)^{-\frac{1+5\alpha+18\alpha\beta}{\alpha(\alpha+5)+2\beta\{3+4\alpha(1+2\beta)\}}}$$

$$\text{and } f(z) = z(1+2z^2)^{\frac{1}{1+5\alpha+18\alpha\beta}}.$$

COROLLARY-2:

$TK[1,0] = K$, as by substituting $\alpha = 1$ and $\beta = 0$ the result becomes:

$$|a_3 - \mu a_2^2| \leq \begin{cases} 1 - \mu, & \text{if } \mu \leq \frac{2}{3}; \\ \frac{1}{3}, & \text{if } \frac{2}{3} \leq \mu \leq \frac{4}{3}; \\ \mu - 1, & \text{if } \mu \geq \frac{4}{3}. \end{cases}$$

which is the required result for the class K given by Keogh and Merkes [5].

THEOREM3:

Let $f(z) \in TK[\alpha, \beta, \delta]$ and $\phi(z) = \frac{1+w(z)}{1-w(z)}$; $w(z)$ is a Schwarzian function, then:

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{16\alpha(1+2\beta)\delta^2 + 2\delta^2[1+\alpha(1+4\beta)]^2}{[1+\alpha(1+4\beta)]^2(1+5\alpha+18\alpha\beta)} - \frac{4\mu\delta^2}{[1+\alpha(1+4\beta)]^2}; \\ \mu \leq \frac{8\alpha(1+2\beta)\delta^2 + (\delta^2 - \delta)[1+\alpha(1+4\beta)]^2}{1+5\alpha+18\alpha\beta}; \\ \frac{2\delta}{1+5\alpha+18\alpha\beta}; \frac{8\alpha(1+2\beta)\delta^2 + (\delta^2 - \delta)[1+\alpha(1+4\beta)]^2}{1+5\alpha+18\alpha\beta} \leq \\ \mu \leq \frac{8\alpha(1+2\beta)\delta^2 + (\delta^2 + \delta)[1+\alpha(1+4\beta)]^2}{1+5\alpha+18\alpha\beta}; \\ \frac{4\mu\delta^2}{[1+\alpha(1+4\beta)]^2} - \frac{16\alpha(1+2\beta)\delta^2 + 2\delta^2[1+\alpha(1+4\beta)]^2}{[1+\alpha(1+4\beta)]^2(1+5\alpha+18\alpha\beta)}; \\ \mu \geq \frac{8\alpha(1+2\beta)\delta^2 + (\delta^2 + \delta)[1+\alpha(1+4\beta)]^2}{1+5\alpha+18\alpha\beta}. \end{cases}$$

PROOF:

By definition of $TK[\alpha, \beta, \delta]$,

$$(1-\alpha)\frac{f(z)}{z} + \alpha \frac{\{zf'(z) + \beta z^2 f''(z)\}'}{\{f(z)\}'} = \left(\frac{1+w(z)}{1-w(z)} \right)^\delta \quad (6)$$

where

$$w(z) = c_1 z + c_2 z^2 + c_3 z^3 +$$

$$f(z) = z + a_2 z^2 + a_3 z^3 +$$

$$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3$$

$$f''(z) = 2a_2 + 6a_3 z + 12a_4 z^2$$

Now by putting all these values in (6), we get:

$$(1-\alpha)(1+a_2z + a_3z^2 + \dots) + \alpha \left(\frac{\{z+2a_2z^2+3a_3z^3+\beta 2 a_2z^2+\beta 6a_3z^3+\beta 12 a_4z^4+\dots\}'}{(1+2a_2z+3a_3z^2+\dots)} \right) = \left(\frac{1+c_1z+c_2z^2+\dots}{1-(c_1z+c_2z^2+\dots)} \right)^\delta$$

By expanding the series, we get:

$$1 + [1+\alpha(1+4\beta)] a_2z + [\{1 + \alpha(5 + 18\beta)\}a_3 - 4\alpha(1 + 2\beta)a_2^2]z^2 + \dots = 1 + 2\delta c_1z + 2(\delta c_2 + \delta^2 c_1^2)z^2 + \dots$$

By comparing, we get:

$$a_2 = \frac{2\delta c_1}{1+\alpha(1+4\beta)}$$

and

$$a_3 = \frac{2[1 + \alpha(1 + 4\beta)]^2 \delta c_2 + [16\alpha(1 + 2\beta) + 2\{1 + \alpha(1 + 4\beta)\}^2] \delta^2 c_1^2}{\{1 + \alpha(5 + 18\beta)\}\{1 + \alpha(1 + 4\beta)\}^2}$$

Using these values of a_2 and a_3 , we get:

$$a_3 - \mu a_2^2 = \frac{2\delta c_2}{\{1+\alpha(5+18\beta)\}} + \left(\frac{16\alpha(1+2\beta)+2\{1+\alpha(1+4\beta)\}^2}{\{1+\alpha(5+18\beta)\}\{1+\alpha(1+4\beta)\}^2} - \frac{4\mu}{\{1+\alpha(1+4\beta)\}^2} \right) \delta^2 c_1^2$$

After applying mode on both sides, we get:

$$|a_3 - \mu a_2^2| \leq \left(\frac{2\delta}{\{1+\alpha(5+18\beta)\}} \right) |c_2| + \left| \left(\frac{16\alpha(1+2\beta)+2\{1+\alpha(1+4\beta)\}^2}{\{1+\alpha(5+18\beta)\}\{1+\alpha(1+4\beta)\}^2} - \frac{4\mu}{\{1+\alpha(1+4\beta)\}^2} \right) \delta^2 \right| |c_1|^2$$

Using $|c_2| \leq 1 - |c_1|^2$, we get:

$$|a_3 - \mu a_2^2| \leq \frac{2\delta}{\{1+\alpha(5+18\beta)\}} + \left\{ \left| \left(\frac{16\alpha(1+2\beta)+2\{1+\alpha(1+4\beta)\}^2}{\{1+\alpha(5+18\beta)\}\{1+\alpha(1+4\beta)\}^2} - \frac{4\mu}{\{1+\alpha(1+4\beta)\}^2} \right) \delta^2 \right| - \frac{2\delta}{\{1+\alpha(5+18\beta)\}} \right\} |c_1|^2$$

Case 1:

If $\mu \leq \frac{8\alpha(1+2\beta)+\{1+\alpha(1+4\beta)\}^2}{2\{1+\alpha(5+18\beta)\}}$, then,

$$|a_3 - \mu a_2^2| \leq \frac{2\delta}{\{1+\alpha(5+18\beta)\}} + \left\{ \frac{16\alpha(1+2\beta)\delta^2+2(\delta^2-\delta)\{1+\alpha(1+4\beta)\}^2}{\{1+\alpha(5+18\beta)\}\{1+\alpha(1+4\beta)\}^2} - \frac{4\mu\delta^2}{\{1+\alpha(1+4\beta)\}^2} \right\} |c_1|^2$$

Subcase – 1 (a):

When

$$\mu \leq \frac{8\alpha(1+2\beta)\delta^2+(\delta^2-\delta)\{1+\alpha(1+4\beta)\}^2}{2\delta^2\{1+\alpha(5+18\beta)\}}$$

Then, by using $|c_1| \leq 1$, we get:

$$|a_3 - \mu a_2^2| \leq \frac{2\delta^2\{1+\alpha(1+4\beta)\}^2+16\alpha(1+2\beta)\delta^2}{\{1+\alpha(5+18\beta)\}\{1+\alpha(1+4\beta)\}^2} - \frac{4\mu\delta^2}{\{1+\alpha(1+4\beta)\}^2} \quad (7)$$

Subcase – 1 (b):

When $\mu \geq \frac{8\alpha(1+2\beta)\delta^2+(\delta^2-\delta)\{1+\alpha(1+4\beta)\}^2}{2\delta^2\{1+\alpha(5+18\beta)\}}$, then,

$$|a_3 - \mu a_2^2| \leq \frac{2\delta}{\{1+\alpha(5+18\beta)\}} \quad (8)$$

Case – 2:

If $\mu \geq \frac{8\alpha(1+2\beta)+\{1+\alpha(1+4\beta)\}^2}{2\{1+\alpha(5+18\beta)\}}$, then,

$$|a_3 - \mu a_2^2| \leq \frac{2\delta}{\{1+\alpha(5+18\beta)\}} + \left\{ \frac{4\mu\delta^2}{\{1+\alpha(1+4\beta)\}^2} - \frac{16\alpha(1+2\beta)\delta^2+2(\delta^2+\delta)\{1+\alpha(1+4\beta)\}^2}{\{1+\alpha(5+18\beta)\}\{1+\alpha(1+4\beta)\}^2} \right\} |c_1|^2$$

Subcase-2 (a):

When $\mu \geq \frac{8\alpha(1+2\beta)\delta^2 + (\delta^2 + \delta)\{1 + \alpha(1+4\beta)\}^2}{2\delta^2\{1 + \alpha(5+18\beta)\}}$, then,

$$|a_3 - \mu a_2^2| \leq -\frac{2\delta^2\{1 + \alpha(1+4\beta)\}^2 + 16\alpha(1+2\beta)\delta^2}{\{1 + \alpha(5+18\beta)\}\{1 + \alpha(1+4\beta)\}^2} + \frac{4\mu\delta^2}{\{1 + \alpha(1+4\beta)\}^2} \quad (9)$$

Subcase – 2 (b):

When $\mu \leq \frac{8\alpha(1+2\beta)\delta^2 + (\delta^2 + \delta)\{1 + \alpha(1+4\beta)\}^2}{2\delta^2\{1 + \alpha(5+18\beta)\}}$, then,

$$|a_3 - \mu a_2^2| \leq \frac{2\delta}{\{1 + \alpha(5+18\beta)\}} \quad (10)$$

Combining (7), (8), (9) and (10), we get the required result.

EXTREMALS:

The result is sharp for extremal functions:

$$f(z) = z \left(1 - \frac{2\delta\alpha[(\alpha + 5) + 2\beta\{3 + 4\alpha(1 + 2\beta)\}]}{[1 + \alpha(1 + 4\beta)][1 + (5 + 18\beta)\alpha]} z \right)^{-\frac{1+5\alpha+18\alpha\beta}{\alpha[(5+\alpha)+2\beta\{3+4\alpha(1+2\beta)\}]}}$$

$$\text{and } f(z) = z(1 + 2\delta z^2)^{\frac{1}{1+\alpha(5+18\beta)}}.$$

COROLLARY-4:

$TK[\alpha, \beta, 1] = TK[\alpha, \beta]$, as by substituting $\delta = 1$, the result becomes:

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{16\alpha(1+2\beta)+2[1+\alpha(1+4\beta)]^2}{[1+\alpha(1+4\beta)]^2[1+(5+18\beta)\alpha]} - \frac{4\mu}{[1+\alpha(1+4\beta)]^2}; \\ \mu \leq \frac{4\alpha(1+2\beta)}{1+\alpha(5+18\beta)}; \\ \frac{2}{1+5\alpha+18\alpha\beta}; \\ \frac{4\alpha(1+2\beta)}{1+\alpha(5+18\beta)} \leq \mu \leq \frac{4\alpha(1+2\beta)+[1+\alpha(1+4\beta)]^2}{1+\alpha(5+18\beta)}; \\ \frac{4\mu}{[1+\alpha(1+4\beta)]^2} - \frac{16\alpha(1+2\beta)+2[1+\alpha(1+4\beta)]^2}{[1+\alpha(1+4\beta)]^2[1+\alpha(5+18\beta)]}; \\ \mu \geq \frac{4\alpha(1+2\beta)+[1+\alpha(1+4\beta)]^2}{1+\alpha(5+18\beta)}. \end{cases}$$

which is the required result for the class $TK[\alpha, \beta]$.

COROLLARY-5:

$TK[1, 0, 1] = K$, as by substituting $\alpha = 1$, $\beta = 0$ and $\delta = 1$, the result becomes:

$$|a_3 - \mu a_2^2| \leq \begin{cases} 1 - \mu, \text{ if } \mu \leq \frac{2}{3}; \\ \frac{1}{3}, \text{ if } \frac{2}{3} \leq \mu \leq \frac{4}{3}; \\ \mu - 1, \text{ if } \mu \geq \frac{4}{3}. \end{cases}$$

which is the required result for the class K .

THEOREM6:

Let $f(z) \in TK[\alpha, \beta, A, B]$ and $\phi(z) = \frac{1+Aw(z)}{1+Bw(z)}$; $w(z)$ is a Schwarzian function, then:

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{4\alpha(1+2\beta)(A-B)^2 - B(A-B)[1+\alpha(1+4\beta)]^2}{(1+5\alpha+18\alpha\beta)[1+\alpha(1+4\beta)]^2} - \frac{\mu(A-B)^2}{[1+\alpha(1+4\beta)]^2}; \\ \mu \leq \frac{4\alpha(1+2\beta)(A-B) - (B+1)[1+\alpha(1+4\beta)]^2}{(A-B)(1+5\alpha+18\alpha\beta)}; \\ \frac{A-B}{1+5\alpha+18\alpha\beta} - \frac{4\alpha(1+2\beta)(A-B) - (B+1)[1+\alpha(1+4\beta)]^2}{(A-B)(1+5\alpha+18\alpha\beta)} \leq \mu \\ \leq \frac{4\alpha(1+2\beta)(A-B) + (1-B)[1+\alpha(1+4\beta)]^2}{(A-B)(1+5\alpha+18\alpha\beta)}; \\ \frac{\mu(A-B)^2}{[1+\alpha(1+4\beta)]^2} - \frac{4\alpha(1+2\beta)(A-B)^2 - B(A-B)[1+\alpha(1+4\beta)]^2}{(1+5\alpha+18\alpha\beta)[1+\alpha(1+4\beta)]^2}; \\ \mu \geq \frac{4\alpha(1+2\beta)(A-B) + (1-B)[1+\alpha(1+4\beta)]^2}{(A-B)(1+5\alpha+18\alpha\beta)}. \end{cases}$$

PROOF:

By definition of $TK[\alpha, \beta, A, B]$,

$$(1-\alpha)\frac{f(z)}{z} + \alpha \frac{\{zf'(z) + \beta z^2 f''(z)\}'}{\{f(z)\}'} = \frac{1+Aw(z)}{1+Bw(z)} \quad (11)$$

where

$$w(z) = c_1 z + c_2 z^2 + c_3 z^3$$

$$f(z) = z + a_2 z^2 + a_3 z^3$$

$$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3$$

$$f''(z) = 2a_2 + 6a_3 z + 12a_4 z^2$$

Now by putting all these values in (11), we get:

$$(1-\alpha)(1+a_2 z + a_3 z^2) + \alpha \left(\frac{\{z+2a_2 z^2+3a_3 z^3+\beta 2 a_2 z^2+\beta 6 a_3 z^3+\beta 12 a_4 z^4+\dots\}'}{(1+2a_2 z+3a_3 z^2+\dots)} \right) = \frac{1+A(c_1 z + c_2 z^2 + \dots)}{1+B(c_1 z + c_2 z^2 + \dots)}$$

By expanding the series, we get:

$$1 + [1+\alpha(1+4\beta)] a_2 z + [\{1+\alpha(5+18\beta)\} a_3 - 4\alpha(1+2\beta)a_2^2] z^2 = 1 + (A-B)c_1 z + [(A-B)c_2 - B(A-B)c_1^2] z^2$$

By comparing, we get:

$$a_2 = \frac{(A-B)c_1}{1+\alpha(1+4\beta)}$$

and

$$a_3 = \frac{(A-B)[1+\alpha(1+4\beta)]^2 c_2 + (A-B)[4\alpha(1+2\beta)(A-B) - B\{1+\alpha(1+4\beta)\}^2] c_1^2}{\{1+\alpha(5+18\beta)\}\{1+\alpha(1+4\beta)\}^2}$$

Using these values of a_2 and a_3 , we get:

$$a_3 - \mu a_2^2 = \frac{(A-B)c_2}{1+\alpha(5+18\beta)} + \left(\frac{[4\alpha(1+2\beta)(A-B) - B\{1+\alpha(1+4\beta)\}^2](A-B)}{\{1+\alpha(5+18\beta)\}\{1+\alpha(1+4\beta)\}^2} - \frac{(A-B)^2 \mu}{\{1+\alpha(1+4\beta)\}^2} \right) c_1^2$$

After applying mode on both sides, we get:

$$|a_3 - \mu a_2^2| \leq \left(\frac{(A-B)}{1+\alpha(5+18\beta)} \right) |c_2| + \left| \frac{[4\alpha(1+2\beta)(A-B) - B\{1+\alpha(1+4\beta)\}^2](A-B)}{\{1+\alpha(5+18\beta)\}\{1+\alpha(1+4\beta)\}^2} - \frac{(A-B)^2 \mu}{\{1+\alpha(1+4\beta)\}^2} \right| |c_1|^2$$

Using $|c_2| \leq 1 - |c_1|^2$, we get:

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)}{1+\alpha(5+18\beta)} + \left\{ \left| \frac{[4\alpha(1+2\beta)(A-B)-B\{1+\alpha(1+4\beta)\}^2](A-B)}{\{1+\alpha(5+18\beta)\}\{1+\alpha(1+4\beta)\}^2} - \frac{(A-B)^2\mu}{\{1+\alpha(1+4\beta)\}^2} \right| - \frac{(A-B)}{1+\alpha(5+18\beta)} \right\} |c_1|^2$$

Case 1:

If $\mu \leq \frac{4\alpha(1+2\beta)(A-B)-B\{1+\alpha(1+4\beta)\}^2}{(A-B)\{1+\alpha(5+18\beta)\}}$, then,

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)}{1+\alpha(5+18\beta)} + \left\{ \frac{[4\alpha(1+2\beta)(A-B)-(B+1)\{1+\alpha(1+4\beta)\}^2](A-B)}{\{1+\alpha(5+18\beta)\}\{1+\alpha(1+4\beta)\}^2} - \frac{(A-B)^2\mu}{\{1+\alpha(1+4\beta)\}^2} \right\} |c_1|^2$$

Subcase – 1 (a):

When $\mu \leq \frac{4\alpha(1+2\beta)(A-B)-(B+1)\{1+\alpha(1+4\beta)\}^2}{(A-B)\{1+\alpha(5+18\beta)\}}$

By using $|c_1| \leq 1$, we get:

$$|a_3 - \mu a_2^2| \leq \frac{[4\alpha(1+2\beta)(A-B)-B\{1+\alpha(1+4\beta)\}^2](A-B)}{\{1+\alpha(5+18\beta)\}\{1+\alpha(1+4\beta)\}^2} - \frac{(A-B)^2\mu}{\{1+\alpha(1+4\beta)\}^2} \quad (12)$$

Subcase – 1 (b):

When $\mu \geq \frac{4\alpha(1+2\beta)(A-B)-(B+1)\{1+\alpha(1+4\beta)\}^2}{(A-B)\{1+\alpha(5+18\beta)\}}$, then,

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)}{1+\alpha(5+18\beta)} \quad (13)$$

Case – 2:

If $\mu \geq \frac{4\alpha(1+2\beta)(A-B)-B\{1+\alpha(1+4\beta)\}^2}{(A-B)\{1+\alpha(5+18\beta)\}}$, then,

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)}{1+\alpha(5+18\beta)} + \left\{ \frac{(A-B)^2\mu}{\{1+\alpha(1+4\beta)\}^2} - \frac{[4\alpha(1+2\beta)(A-B)+(1-B)\{1+\alpha(1+4\beta)\}^2](A-B)}{\{1+\alpha(5+18\beta)\}\{1+\alpha(1+4\beta)\}^2} \right\} |c_1|^2$$

Subcase-2 (a):

When $\mu \geq \frac{4\alpha(1+2\beta)(A-B)+(1-B)\{1+\alpha(1+4\beta)\}^2}{(A-B)\{1+\alpha(5+18\beta)\}}$, then,

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)^2\mu}{\{1+\alpha(1+4\beta)\}^2} - \frac{[4\alpha(1+2\beta)(A-B)-B\{1+\alpha(1+4\beta)\}^2](A-B)}{\{1+\alpha(5+18\beta)\}\{1+\alpha(1+4\beta)\}^2} \quad (14)$$

Subcase – 2 (b):

When $\mu \leq \frac{4\alpha(1+2\beta)(A-B)+(1-B)\{1+\alpha(1+4\beta)\}^2}{(A-B)\{1+\alpha(5+18\beta)\}}$, then,

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)}{1+\alpha(5+18\beta)} \quad (15)$$

Combining (12), (13), (14) and (15), we get the required result.

EXTREMALS:

Extremal functions of this inequality is given by:

$$f(z)=z \left(1 + \frac{\frac{A(1-3\alpha)+B(1+2\alpha^2+7\alpha)+2\alpha\beta[A+B\{7+8\alpha(1+2\beta)\}]}{\{1+\alpha(5+18\beta)\}\{1+\alpha(1+4\beta)\}} z}{\frac{(1+5\alpha+18\alpha\beta)(A-B)}{A(1-3\alpha)+B(1+2\alpha^2+7\alpha)+2\alpha\beta[A+B\{7+8\alpha(1+2\beta)\}]}} \right)$$

and $f(z) = z [1 + (A-B)z^2]^{\frac{1}{1+\alpha(5+18\beta)}}$.

COROLLARY-7:

$TK[\alpha, \beta, 1, -1] = TK[\alpha, \beta]$, as by substituting $A=1$ and $B=-1$, the result becomes:

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{16\alpha(1+2\beta)+2[1+\alpha(1+4\beta)]^2}{[1+\alpha(1+4\beta)]^2(1+5\alpha+18\alpha\beta)} - \frac{4\mu}{[1+\alpha(1+4\beta)]^2}; \mu \leq \frac{4\alpha(1+2\beta)}{1+5\alpha+18\alpha\beta}; \\ \frac{2}{1+5\alpha+18\alpha\beta}; \frac{4\alpha(1+2\beta)}{1+5\alpha+18\alpha\beta} \leq \mu \leq \frac{4\alpha(1+2\beta)+[1+\alpha(1+4\beta)]^2}{(1+5\alpha+18\alpha\beta)}; \\ \frac{4\mu}{[1+\alpha(1+4\beta)]^2} - \frac{16\alpha(1+2\beta)+2[1+\alpha(1+4\beta)]^2}{[1+\alpha(1+4\beta)]^2(1+5\alpha+18\alpha\beta)}; \mu \geq \frac{4\alpha(1+2\beta)+[1+\alpha(1+4\beta)]^2}{(1+5\alpha+18\alpha\beta)}. \end{cases}$$

which is the required result for the class $TK[\alpha, \beta]$.

COROLLARY-8:

$TK[1,0,1,-1]=K$, as by substituting $\alpha=1, \beta=0, A=1$ and $B=-1$, the result becomes:

$$|a_3 - \mu a_2^2| \leq \begin{cases} 1 - \mu, \text{ if } \mu \leq \frac{2}{3}; \\ \frac{1}{3}, \text{ if } \frac{2}{3} \leq \mu \leq \frac{4}{3}; \\ \mu - 1, \text{ if } \mu \geq \frac{4}{3}. \end{cases}$$

which is the required result for the class K .

THEOREM9:

If $f(z) \in TK[\alpha, \beta, A, B, \delta]$ and $\phi(z) = \left(\frac{1+Aw(z)}{1+Bw(z)}\right)^\delta$, then:

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\delta(A-B) \left\{ \left[\frac{\delta}{2}(A-B) - \frac{1}{2}(A+B) \right] [1+\alpha(1+4\beta)]^2 + 4\alpha\delta(A-B)(1+2\beta) \right\}}{[1+\alpha(1+4\beta)]^2(1+5\alpha+18\alpha\beta)} - \mu \frac{\delta^2(A-B)^2}{[1+\alpha(1+4\beta)]^2}; \\ \mu \leq \frac{\left\{ \frac{\delta}{2}(A-B) - \frac{1}{2}(A+B) - 1 \right\} [1+\alpha(1+4\beta)]^2 + 4\alpha\delta(A-B)(1+2\beta)}{\delta(A-B)(1+5\alpha+18\alpha\beta)}; \\ \frac{\delta(A-B)}{1+5\alpha+18\alpha\beta}; \frac{\left\{ \frac{\delta}{2}(A-B) - \frac{1}{2}(A+B) - 1 \right\} [1+\alpha(1+4\beta)]^2 + 4\alpha\delta(A-B)(1+2\beta)}{\delta(A-B)(1+5\alpha+18\alpha\beta)} \leq \mu \leq \frac{\left\{ \frac{\delta}{2}(A-B) - \frac{1}{2}(A+B) + 1 \right\} [1+\alpha(1+4\beta)]^2 + 4\alpha\delta(A-B)(1+2\beta)}{\delta(A-B)(1+5\alpha+18\alpha\beta)}; \\ \mu \geq \frac{\delta(A-B) \left\{ \left[\frac{\delta}{2}(A-B) - \frac{(A+B)}{2} \right] [1+\alpha(1+4\beta)]^2 + 4\alpha\delta(A-B)(1+2\beta) \right\}}{[1+\alpha(1+4\beta)]^2(1+5\alpha+18\alpha\beta)} - \mu \frac{\delta^2(A-B)^2}{[1+\alpha(1+4\beta)]^2}; \\ \mu \geq \frac{\left\{ \frac{\delta}{2}(A-B) - \frac{1}{2}(A+B) + 1 \right\} [1+\alpha(1+4\beta)]^2 + 4\alpha\delta(A-B)(1+2\beta)}{\delta(A-B)(1+5\alpha+18\alpha\beta)}. \end{cases}$$

PROOF:

By definition of $TK[\alpha, \beta, A, B, \delta]$,

$$(1-\alpha) \frac{f(z)}{z} + \alpha \frac{\{zf'(z) + \beta z^2 f''(z)\}'}{\{f(z)\}'} = \left(\frac{1+Aw(z)}{1+Bw(z)}\right)^\delta \quad (16)$$

where

$$w(z) = c_1 z + c_2 z^2 + c_3 z^3$$

$$f(z) = z + a_2 z^2 + a_3 z^3$$

$$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3$$

$$f''(z) = 2a_2 + 6a_3 z + 12a_4 z^2$$

Now, by putting all these values in (16), we get:

$$(1 - \alpha) (1 + a_2 z + a_3 z^2 + \dots) + \alpha \left(\frac{\{z + 2a_2 z^2 + 3a_3 z^3 + \beta 2 a_2 z^2 + \beta 6 a_3 z^3 + \beta 12 a_4 z^4 + \dots\}'}{(1 + 2a_2 z + 3a_3 z^2 + \dots)} \right) \\ = \left(\frac{1 + A(c_1 z + c_2 z^2 + \dots)}{1 + B(c_1 z + c_2 z^2 + \dots)} \right)^\delta$$

By expanding the series, we get:

$$1 + [1 + \alpha(1 + 4\beta)] a_2 z + [\{1 + \alpha(5 + 18\beta)\} a_3 - 4\alpha(1 + 2\beta) a_2^2] z^2 \\ = 1 + \delta(A - B) c_1 z + [\delta(A - B) c_2 + \frac{\delta^2}{2} (A^2 + B^2 - 2AB) c_1^2 - \frac{\delta}{2} (A^2 - B^2) c_1^2] z^2.$$

By comparing, we get:

$$a_2 = \frac{\delta(A - B) c_1}{1 + \alpha(1 + 4\beta)}$$

and

$$\delta(A - B) [1 + \alpha(1 + 4\beta)]^2 c_2 + \delta(A - B) [4\alpha\delta(1 + 2\beta)(A - B) + \\ \frac{1}{2} \{\delta(A - B) - (A + B)\} \{1 + \alpha(1 + 4\beta)\}^2] c_1^2 \\ a_3 = \frac{\delta(A - B) [4\alpha\delta(1 + 2\beta)(A - B) + \frac{1}{2} \{\delta(A - B) - (A + B)\} \{1 + \alpha(1 + 4\beta)\}^2] c_1^2}{\{1 + \alpha(5 + 18\beta)\} \{1 + \alpha(1 + 4\beta)\}^2}$$

Using these values of a_2 and a_3 , we get:

$$a_3 - \mu a_2^2 = \frac{\delta(A - B) c_2}{1 + \alpha(5 + 18\beta)} + \left(\frac{\delta(A - B) [4\alpha\delta(1 + 2\beta)(A - B) + \frac{1}{2} \{\delta(A - B) - (A + B)\} \{1 + \alpha(1 + 4\beta)\}^2]}{\{1 + \alpha(5 + 18\beta)\} \{1 + \alpha(1 + 4\beta)\}^2} - \frac{\delta^2 (A - B)^2 \mu}{\{1 + \alpha(1 + 4\beta)\}^2} \right) c_1^2$$

After applying mode on both sides,

we get

$$|a_3 - \mu a_2^2| \leq \left(\frac{\delta(A - B)}{1 + \alpha(5 + 18\beta)} \right) |c_2| + \left| \frac{\delta(A - B) [4\alpha\delta(1 + 2\beta)(A - B) + \frac{1}{2} \{\delta(A - B) - (A + B)\} \{1 + \alpha(1 + 4\beta)\}^2]}{\{1 + \alpha(5 + 18\beta)\} \{1 + \alpha(1 + 4\beta)\}^2} - \frac{\delta^2 (A - B)^2 \mu}{\{1 + \alpha(1 + 4\beta)\}^2} \right| |c_1|^2$$

Using $|c_2| \leq 1 - |c_1|^2$, we get:

$$|a_3 - \mu a_2^2| \leq \frac{\delta(A - B)}{1 + \alpha(5 + 18\beta)} + \left\{ \left| \frac{\delta(A - B) [4\alpha\delta(1 + 2\beta)(A - B) + \frac{1}{2} \{\delta(A - B) - (A + B)\} \{1 + \alpha(1 + 4\beta)\}^2]}{\{1 + \alpha(5 + 18\beta)\} \{1 + \alpha(1 + 4\beta)\}^2} - \frac{\delta^2 (A - B)^2 \mu}{\{1 + \alpha(1 + 4\beta)\}^2} \right| - \right. \\ \left. \frac{\delta(A - B)}{1 + \alpha(5 + 18\beta)} \right\} |c_1|^2$$

Case 1:

If $\mu \leq \frac{[4\alpha\delta(1 + 2\beta)(A - B) + \frac{1}{2} \{\delta(A - B) - (A + B)\} \{1 + \alpha(1 + 4\beta)\}^2]}{\delta(A - B) \{1 + \alpha(5 + 18\beta)\}}$, then,

$$|a_3 - \mu a_2^2| \leq \frac{\delta(A - B)}{1 + \alpha(5 + 18\beta)} + \left\{ \frac{\delta(A - B) [4\alpha\delta(1 + 2\beta)(A - B) + \frac{\delta}{2} (A - B) - \frac{1}{2} (A + B) - 1] \{1 + \alpha(1 + 4\beta)\}^2}{\{1 + \alpha(5 + 18\beta)\} \{1 + \alpha(1 + 4\beta)\}^2} - \frac{\delta^2 (A - B)^2 \mu}{\{1 + \alpha(1 + 4\beta)\}^2} \right\} |c_1|^2$$

Subcase – 1 (a):

When $\mu \leq \frac{4\alpha\delta(1 + 2\beta)(A - B) + \frac{\delta}{2} (A - B) - \frac{1}{2} (A + B) - 1}{\delta(A - B) \{1 + \alpha(5 + 18\beta)\}}$, then, by using $|c_1| \leq 1$, we get:

$$|a_3 - \mu a_2^2| \leq \frac{\delta(A - B) [4\alpha\delta(1 + 2\beta)(A - B) + \frac{\delta}{2} (A - B) - \frac{1}{2} (A + B)] \{1 + \alpha(1 + 4\beta)\}^2}{\{1 + \alpha(5 + 18\beta)\} \{1 + \alpha(1 + 4\beta)\}^2} - \frac{\delta^2 (A - B)^2 \mu}{\{1 + \alpha(1 + 4\beta)\}^2} \quad (17)$$

Subcase – 1 (b):

When $\mu \geq \frac{4\alpha\delta(1+2\beta)(A-B) + \left\{\frac{\delta}{2}(A-B) - \frac{1}{2}(A+B) - 1\right\}\{1+\alpha(1+4\beta)\}^2}{\delta(A-B)\{1+\alpha(5+18\beta)\}}$, then,

$$|a_3 - \mu a_2^2| \leq \frac{\delta(A-B)}{1+\alpha(5+18\beta)} \quad (18)$$

Case – 2:

If $\mu \geq \frac{[4\alpha\delta(1+2\beta)(A-B) + \frac{1}{2}\{\delta(A-B) - (A+B)\}\{1+\alpha(1+4\beta)\}^2]}{\delta(A-B)\{1+\alpha(5+18\beta)\}}$, then,

$$|a_3 - \mu a_2^2| \leq \frac{\delta(A-B)}{1+\alpha(5+18\beta)} + \left\{ \frac{\delta^2(A-B)^2\mu}{\{1+\alpha(1+4\beta)\}^2} - \frac{\delta(A-B)[4\alpha\delta(1+2\beta)(A-B) + \left\{\frac{\delta}{2}(A-B) - \frac{1}{2}(A+B) + 1\right\}\{1+\alpha(1+4\beta)\}^2]}{\{1+\alpha(5+18\beta)\}\{1+\alpha(1+4\beta)\}^2} \right\} |c_1|^2$$

Subcase-2 (a):

When $\mu \geq \frac{4\alpha\delta(1+2\beta)(A-B) + \left\{\frac{\delta}{2}(A-B) - \frac{1}{2}(A+B) + 1\right\}\{1+\alpha(1+4\beta)\}^2}{\delta(A-B)\{1+\alpha(5+18\beta)\}}$, then,

$$|a_3 - \mu a_2^2| \leq \frac{\delta^2(A-B)^2\mu}{\{1+\alpha(1+4\beta)\}^2} - \frac{\delta(A-B)[4\alpha\delta(1+2\beta)(A-B) + \left\{\frac{\delta}{2}(A-B) - \frac{1}{2}(A+B)\right\}\{1+\alpha(1+4\beta)\}^2]}{\{1+\alpha(5+18\beta)\}\{1+\alpha(1+4\beta)\}^2} \quad (19)$$

Subcase – 2 (b):

When $\mu \leq \frac{4\alpha\delta(1+2\beta)(A-B) + \left\{\frac{\delta}{2}(A-B) - \frac{1}{2}(A+B) + 1\right\}\{1+\alpha(1+4\beta)\}^2}{\delta(A-B)\{1+\alpha(5+18\beta)\}}$, then,

$$|a_3 - \mu a_2^2| \leq \frac{\delta(A-B)}{1+\alpha(5+18\beta)} \quad (20)$$

By combining (17), (18), (19) and (20), we get the required result.

FOR EXTREMALS:

Extremal functions of this inequality is given by:

$$f(z) = z \left(1 + \frac{\frac{\delta(A-B)(1-3\alpha+2\alpha\beta) - [1+\alpha(1+4\beta)]^2}{\{\delta(A-B) - (A+B)\}}}{(1+5\alpha+18\alpha\beta)[1+\alpha(1+4\beta)]^2} z \right)^{\frac{\delta(A-B)(1+5\alpha+18\alpha\beta)}{\delta(A-B)(1-3\alpha+2\alpha\beta)[1+\alpha(1+4\beta)]^2 - \{\delta(A-B) - (A+B)\}}}$$

and $f(z) = z[1 + \delta(A-B)z^2]^{\frac{1}{1+5\alpha+18\alpha\beta}}$.

COROLLARY-10:

$TK[\alpha, \beta, 1, -1, 1] = TK[\alpha, \beta]$, as by substituting $A=1, B=-1$ and $\delta=1$, the result becomes:

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{16\alpha(1+2\beta)+2[1+\alpha(1+4\beta)]^2}{[1+\alpha(1+4\beta)]^2(1+5\alpha+18\alpha\beta)} - \frac{4\mu}{[1+\alpha(1+4\beta)]^2}; \mu \leq \frac{4\alpha(1+2\beta)}{1+5\alpha+18\alpha\beta}; \\ \frac{2}{1+5\alpha+18\alpha\beta}; \frac{4\alpha(1+2\beta)}{1+5\alpha+18\alpha\beta} \leq \mu \leq \frac{4\alpha(1+2\beta)+[1+\alpha(1+4\beta)]^2}{(1+5\alpha+18\alpha\beta)}; \\ \frac{4\mu}{[1+\alpha(1+4\beta)]^2} - \frac{16\alpha(1+2\beta)+2[1+\alpha(1+4\beta)]^2}{[1+\alpha(1+4\beta)]^2(1+5\alpha+18\alpha\beta)}; \\ \mu \geq \frac{4\alpha(1+2\beta)+[1+\alpha(1+4\beta)]^2}{(1+5\alpha+18\alpha\beta)}. \end{cases}$$

which is the required result for the class $TK[\alpha, \beta]$.

COROLLARY-11:

$TK[\alpha, \beta, 1, -1, \delta] = TK[\alpha, \beta, \delta]$, as by substituting $A=1$ and $B=-1$, the result becomes:

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{16\alpha(1+2\beta)\delta^2 + 2\delta^2[1 + \alpha(1+4\beta)]^2}{[1 + \alpha(1+4\beta)]^2(1+5\alpha+18\alpha\beta)} - \frac{4\mu\delta^2}{[1 + \alpha(1+4\beta)]^2}; \\ \mu \leq \frac{8\alpha(1+2\beta)\delta^2 + (\delta^2 - \delta)[1 + \alpha(1+4\beta)]^2}{1+5\alpha+18\alpha\beta}; \\ \frac{2\delta}{1+5\alpha+18\alpha\beta}; \frac{8\alpha(1+2\beta)\delta^2 + (\delta^2 - \delta)[1 + \alpha(1+4\beta)]^2}{1+5\alpha+18\alpha\beta} \leq \mu \\ \leq \frac{8\alpha(1+2\beta)\delta^2 + (\delta^2 + \delta)[1 + \alpha(1+4\beta)]^2}{1+5\alpha+18\alpha\beta}; \\ \frac{4\mu\delta^2}{[1 + \alpha(1+4\beta)]^2} - \frac{16\alpha(1+2\beta)\delta^2 + 2\delta^2[1 + \alpha(1+4\beta)]^2}{[1 + \alpha(1+4\beta)]^2(1+5\alpha+18\alpha\beta)}; \\ \mu \geq \frac{8\alpha(1+2\beta)\delta^2 + (\delta^2 + \delta)[1 + \alpha(1+4\beta)]^2}{1+5\alpha+18\alpha\beta}. \end{cases}$$

which is the required result for the class $TK[\alpha, \beta, \delta]$.

COROLLARY-12:

$TK[\alpha, \beta, A, B, 1] = TK[\alpha, \beta, A, B]$, as by substituting $\delta = 1$, the result becomes:

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{4\alpha(1+2\beta)(A-B)^2 - B(A-B)[1 + \alpha(1+4\beta)]^2}{(1+5\alpha+18\alpha\beta)[1 + \alpha(1+4\beta)]^2} - \frac{\mu(A-B)^2}{[1 + \alpha(1+4\beta)]^2}; \\ \mu \leq \frac{4\alpha(1+2\beta)(A-B) - (B+1)[1 + \alpha(1+4\beta)]^2}{(A-B)(1+5\alpha+18\alpha\beta)}; \\ \frac{A-B}{1+5\alpha+18\alpha\beta}; \frac{4\alpha(1+2\beta)(A-B) - (B+1)[1 + \alpha(1+4\beta)]^2}{(A-B)(1+5\alpha+18\alpha\beta)} \leq \\ \mu \leq \frac{4\alpha(1+2\beta)(A-B) + (1-B)[1 + \alpha(1+4\beta)]^2}{(A-B)(1+5\alpha+18\alpha\beta)}; \\ \frac{\mu(A-B)^2}{[1 + \alpha(1+4\beta)]^2} - \frac{4\alpha(1+2\beta)(A-B)^2 - B(A-B)[1 + \alpha(1+4\beta)]^2}{(1+5\alpha+18\alpha\beta)[1 + \alpha(1+4\beta)]^2}; \\ \mu \geq \frac{4\alpha(1+2\beta)(A-B) + (1-B)[1 + \alpha(1+4\beta)]^2}{(A-B)(1+5\alpha+18\alpha\beta)}. \end{cases}$$

which is the required result for the class $TK[\alpha, \beta, A, B]$.

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