New Series for Construction of Second Order Rotatable Designs

N. Ch. Bhatra Charyulu, A. Saheb Shaik, and G. Jayasree

Abstract — Second order Rotatable designs have good significance in response surface methodology. In this paper, two new series for the construction the same using Binary Ternary Designs were presented with illustrated examples.

Keywords — Balanced incomplete block design, binary ternary design, second order rotatable design.

I. INTRODUCTION

The variance of estimated response at the $u^{th}$ design point of a second order response surface design model when $\Sigma x_{ui}x_{uj}^2=N\lambda_2$; $\Sigma x_{ui}x_{uj}^2=CN\lambda_4$; $\Sigma x_{ui}x_{uj}x_{uj}^2=N\lambda_4$ (the summation is over the design points $u = 1, 2, \ldots N$ and all other moments of odd power summations in moment matrix are vanishes, is

$$V(\hat{Y}_u) = V(\hat{\beta}_0) + 2\text{Cov}(\hat{\beta}_0, \hat{\beta}_{ii}) \cdot \rho^2 + V(\hat{\beta}_{ii}) \cdot \rho^4 + \frac{(c-3)}{(c-1)N\lambda_4} \cdot \sum x_{ui}^2 \cdot x_{uj}^2$$

where $\sum x_{ui}^2 = \rho^2$ distance between the point to origin. When $c = 3$ it can be expressed as a function of $\rho^2$ as

$$V(\hat{Y}_u) = \alpha \cdot \rho^2 + \beta \cdot \rho^2 + \gamma.$$

Two new series for the construction of Second Order Rotatable Designs using Balanced Ternary Designs were constructed and presented in the following section.

II. METHODOLOGY

A series of Second Order Rotatable Designs for $v$ factors can be constructed using the Balanced Ternary Designs of Nigam (1974). The detailed procedure is presented below.

Step 1: Let $N_1$ be the incidence matrix of a Balanced Incomplete Block Design with parameters $v, b, r, k, \lambda$. Form the blocks by adding the elements of all possible pairs of

i) $(i, i') (i \leq i')$ of rows of $N_1$, produces a balanced ternary design with parameters $V = v, B = b + ^bC_2, R = r(b+1), K = 2k, \text{and } \pi = (b+2)\lambda + r^2$.

ii) $(i, i') (i < i')$ of rows of $N_1$ (assume $v \geq 2k$), produces a Balanced Ternary Design with parameters $V = v, B = ^bC_2, R = r(b-1), K = 2k, \text{and } \pi = (b-2)\lambda + r^2$.

Step 2: Replace the elements in the balanced ternary design, 2 with $\alpha, 1$ with $\beta$, and associate each block with an appropriate fraction of factorials (say $2^n$) with levels $\pm 1$ so that, no lower order interaction effects are confounded.

Step 3: Add $n_0$ central points $(0, 0, \ldots, 0)$, where $n_0 > 0$ to the resulting design. The numbers of design points are

i) $n = 2^k (b + ^bC_2) + n_0$;

ii) $n = 2^k \cdot ^bC_2 + n_0$.

Step 4: Let $t = \alpha^2 / \beta^2$. Select the real values for ‘t’ as

i) $t = \frac{6(\tau - 1) \pm \sqrt{36}\sqrt{(\tau - 1)^2 - 4(\tau - 1)^2 - 4(\tau - 1)^2 - 3(\tau - 1)^2 + \lambda(\tau - 1)^2 + \lambda(b - 2r)\lambda}\lambda\lambda}}{2(\tau - 1)^2}.$

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ii) $t = \frac{6a(r-\lambda)\sqrt{36a^2(r-\lambda)^2 - 4(c_2-3c_2)[r(b-r) - 3[(r-\lambda)^2 + \lambda(b-2r+\lambda)]]}}{2(c_2-3c_2)}$

Choose the value for ‘$\beta$’, then evaluate the value for ‘$\alpha$’ using $\alpha^2 = t\beta^2$. The resulting design provides a $v$-dimensional Second Order Rotatable Design with five levels ($\pm \alpha, \pm \beta, 0$).

III. FINDINGS

**THEOREM-1**: A Second Order Rotatable Design with five levels ($\pm \alpha, \pm \beta, 0$) can be constructed using Balanced Ternary Design with parameters $V = v$, $B = b + 5C_2$, $R = r(b+1)$, $K = 2k$, and $\pi = (b+2)\alpha + r^2$.

**Proof**: Let $N_{3BV}$ be the incidence matrix of a Balanced Ternary Design with parameters $V = v$, $B = b + 5C_2$, $R = r(b+1)$, $K = 2k$, and $\pi = (b+2)\alpha + r^2$. Each column of the incidence matrix has the elements $2$’s, $1$’s and $0$’s which are repeated $v-1$ $C_2$, $r(b-r)$ and $(b+1)C_2$ times respectively. Every pair of columns contains the pairs $(2,2)$, $(1,1)$ and $(2,1)$ or $(1,2)$ occurs $\lambda$, $[(r-\lambda)^2 + \lambda - b(2r+\lambda)]$ and $2\lambda(\lambda - \lambda)$ times respectively. Replace $C_2$ with $\alpha$ and $1$ with $\beta$. Associate each block with the levels $\pm 1$ by taking an appropriate fraction of factorials (say $2^h$) for $v$ factors. After augmenting $n_0$ central points, the resulting design has $n = 2^h(b + 5C_2)$ and $n_0$ design points.

From the rotatability condition, we obtain

\[
(36a^2(r-\lambda)^2 - 4(c_2-3c_2)[r(b-r) - 3[(r-\lambda)^2 + \lambda(b-2r+\lambda)]] \beta^2 = 0
\]

Solving the quadratic equation in ‘$\beta$’, we have,

\[
t = \frac{6a(r-\lambda)}{\sqrt{36a^2(r-\lambda)^2 - 4(c_2-3c_2)[r(b-r) - 3[(r-\lambda)^2 + \lambda(b-2r+\lambda)]]}}
\]

The roots are real where $9\lambda, 2(r-1)^2 \geq v(c_2-3c_2)[r(b-r) - 3[(r-\lambda)^2 + \lambda(b-2r+\lambda)]]$. Choose any real value for ‘$\beta$’, obtain the level $\alpha$ by choosing from $\alpha^2 = t\beta^2$. The resulting design D provides $v$-dimensional Second Order Rotatable Design in five levels.

**THEOREM-2**: A Second Order Rotatable Design with five levels ($\pm \alpha, \pm \beta, 0$) can be constructed using if there exists Balanced Ternary Design with parameters $V = v$, $B = b + 5C_2$, $R = r(b+1)$, $K = 2k$ and $\pi = (b+2)\lambda + r^2$ then a SORD can be constructed with five levels such that where $\alpha^2 = t\beta^2$ or $\alpha^2 = \beta^2$.

**Proof**: Let $N_{3BV}$ be the incidence matrix of a Balanced Ternary Design with parameters $V = v$, $B = b + 5C_2$, $R = r(b+1)$, $K = 2k$ and $\pi = (b+2)\lambda + r^2$. Each column of the incidence matrix has the elements $2$’s, $1$’s and $0$’s which are repeated $v-1$ $C_2$, $r(b-r)$ and $(b+1)C_2$ times respectively. Every pair of columns contains the pairs $(2,2)$, $(1,1)$ and $(2,1)$ or $(1,2)$ occurs $\lambda$, $[(r-\lambda)^2 + \lambda - b(2r+\lambda)]$ and $2\lambda(\lambda - \lambda)$ times respectively. Replace $C_2$ with $\alpha$ and $1$ with $\beta$. Associate each block with the levels $\pm 1$ by taking an appropriate fraction of factorials (say $2^h$) for $v$ factors. After augmenting $n_0$ central points, the resulting design has $n = 2^h(b + 5C_2)$ and $n_0$ design points. Then we have,

\[
2^h \left[ C_2 \alpha^2 + r(b-r) \beta^2 \right] = \frac{3.2^h}{2} \begin{cases} C_2 \alpha^2 + [r(\lambda-\lambda)^2 + \lambda(b-2r+\lambda)] \beta^2 + 2\lambda(r-\lambda) \alpha^2 \beta^2 \end{cases}
\]

From the rotatable condition, we obtain

\[
(C_2-3c_2)\alpha^2 + [(r(\lambda-\lambda)^2 + \lambda(b-2r+\lambda)) \beta^2 - 6\alpha_2(r-\lambda)(\alpha^2\beta^2) = 0
\]

Let $t = \alpha^2/\beta^2$, then we can obtain

\[
(C_2-3c_2) \alpha^2 + [r(b-r)-3[(r-\lambda)^2 + \lambda(b-2r+\lambda)] = 0
\]

by solving (6), we get $t = \frac{6\lambda \lambda \sqrt{36\lambda^2 (r-\lambda)^2 - 4(c_2-3c_2) (r(b-r) - 3[(r-\lambda)^2 + \lambda(b-2r+\lambda)]}}{2(c_2-3c_2)}$ for the existence of real valued of $t$, i.e.

\[
36\lambda^2 (r-\lambda)^2 \geq 4(c_2-3c_2) (r(b-r) - 3[(r-\lambda)^2 + \lambda(b-2r+\lambda)]
\]

for any real value of ‘$\beta$’, the value ‘$\alpha$’ can be obtained using $\alpha^2 = t\beta^2$. The resulting design D provides a $v$-dimensional Second Order Rotatable Design in five levels.

**EXAMPLE-1**: Let $N_1$ be the incidence matrix of a Balanced Incomplete Block Design with parameters $v = 4$, $b = 6$, $r = 3$, $k = 2$ and $\lambda = 1$ and $N$ be the derived incidence matrix of Balanced Ternary Design.
with parameters V = 4, B = 21, R = 21, K = 4 and π = 17. Then the resulting Second Order Rotatable Design in four factors is presented in Table I.

The real solution for ‘t’ is 4. Select value for β arbitrarily and accordingly α can be evaluated. α² = 4β², when β = ±1 then α = ±1.

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EXAMPLE-2: Let N₁ be the incidence matrix of a Balanced Incomplete Block Design with parameters v = 4, b = 6, r = 3, k = 2 and λ = 1 and N be the derived incidence matrix of Balanced Ternary Design with parameters V = 4, B = 15, R = 15, K = 4 and π = 13. Then the resulting Second Order Rotatable Design in 4 factors is presented in Table II.

The real solution for ‘t’ is 4.4. Select value for β arbitrarily and accordingly α can be evaluated.

<table>
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REFERENCES


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