

The Use of Solvable Directed Graphs in a Jacobi-like Algorithm

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Abstract—In this paper, we introduce a Jacobi-like algorithm (we call D-NJLA) to reduce a real nonsymmetric $n \times n$ matrix to a real upper triangular form by the help of solvable directed graphs. This method uses only real arithmetic and a sequence of orthogonal similarity transformations and achieves ultimate quadratic convergence. A theoretical analysis is constructed and some experimental results are given.

Index Terms—eigenvalues and eigenvectors, Jacobi-like algorithms, transformation methods

I. INTRODUCTION

The numerical computation of matrices' eigenvalues is a problem of major importance in many scientific and engineering applications. The applications areas, where eigenvalue calculations arise, are structural dynamics, electrical networks, macro-economics, quantum chemistry, chemical reactions, control theory, etc [26].

It is generally difficult to determine precisely eigenvalues of a matrix by solving the characteristic equation. So, the fact that similarity transformations on a matrix leave its eigenvalues invariant, i.e., a $n \times n$ matrix can be reduced to the simpler form through unitary similarity transformations

$$\overline{Q}^T A Q = B, \quad (1)$$

where Q is unitary and B is the simpler matrix form (diagonal, upper or lower triangular matrix). So two matrix A and B are called similar. However, as the size of the matrix A increases, finding the unitary similarity matrix Q becomes more difficult. Therefore, the unitary transformation matrix Q is calculated iteratively, i.e.,

$$\overline{Q}_{r+1}^T A_r Q_{r+1} = A_{r+1}, \quad A_0 = A, \quad (2)$$

where Q_{r+1} is the unitary matrix and r is the number of steps [3], [9]. The most well-known of the iterative methods developed for symmetric matrices is the Jacobi method [5], [7], [12], [25]. Because it is successful and easy to implement for the computation of eigenvalues of symmetric matrices, now it has been generalized for nonsymmetric matrices which is known Jacobi-like algorithm [6], [10], [13], [18], [20], [21]. The convergence rate of these algorithm depends on the sorting of the diagonal elements of A and choosing elements to be annihilated. Therefore, in some algorithms, the diagonal elements, are sorted by the choosing of orthogonal transformation matrices and elements to be annihilated [4], [10], [20].

In this paper, we propose a Jacobi-like algorithm, which

computes the Schur form of nonsymmetric square matrices A , with the help of solvable directed graphs, over the real field. The proposed algorithm was designed in two-stages: Stage1 and Stage2. To compute eigenvalues of A with solvable directed graphs, the strategies for choosing the element to be eliminated and the proper orthogonal transformation matrices were determined separately for these stages.

Stage1 is composed of reducing the matrix to nearly 2×2 upper block triangular form by a solvable directed graph. In each step in Stage1, we choose an entry to be eliminated, which has the property that the strictly lower triangular part element has the largest magnitude and the sign of 2×2 subdeterminant is negative, i.e., strictly lower triangular elements whose one diagonal element corresponding to a positive eigenvalue, and another diagonal element corresponding to a negative eigenvalue of the original matrix A are annihilated. Every step of the annihilation, we choose the orthogonal transformation matrix as close as possible to identity and which corresponds to the orthogonal transformation matrix whose rotation parameter falls into $[-3, -2]$ as much as possible.

Stage2 consists of reducing the matrix that is obtained by the result of Stage1 to 2×2 partitioned form whose diagonal blocks are upper triangular matrices whose diagonal elements correspond to greater than or equal to zero and negative eigenvalues of the original matrix A with the help of a solvable directed graph. In each step in Stage2, we choose an entry to be eliminated, which has the property that the strictly lower triangular part element has the largest magnitude and the sign of 2×2 subdeterminant is positive or 2×2 subdeterminant equals to zero, i.e., strictly lower triangular elements whose both diagonal elements corresponding same signed eigenvalues of the original matrix A are annihilated. At every step of the annihilation, we choose the transformation matrix as close as possible to identity and which corresponds to the orthogonal transformation matrix whose rotation parameter falls into $[0, 1]$ as much as possible. The matrix obtained by the result of Stage2 is reduced in Stage1 in the next cycle.

Stage1 and Stage2 have been alternatively applied until reducing the matrix A to the upper triangular form. So, processing in this manner eigenvalues of the matrix A are computed by the help of solvable directed graphs.

The remainder of the paper is organized as follows: the connection between the annihilation process and directed graphs is given in Section 2. In Section 3 the algorithm for calculating eigenvalues is described. We give some numerical experiments in Section 4. The conclusions are given in Section 5.

II. GRAPH THEORETICAL APPROACH TO THE ALGORITHM

The algorithm was built based on some definitions in graph theory. So it would be useful to give some necessary basic graph theory definitions. We consider only finite directed graphs.

Definition 2.1: A directed graph (digraph) is a pair $D = (V, E)$, where V is a set of vertices and $E \subseteq V \times V$ is a binary relation representing the directed edges of D [19]. For a vertex $x \in D$, we denote $E(x) = \{y \in D | E(x, y)\}$ the set successors of x .

Definition 2.2: Let $D = (V, E)$ be a directed graph. A subset of vertices \mathcal{K} is a kernel of D when:

- \mathcal{K} is stable: it contains no pair of adjacent vertices
- \mathcal{K} is absorbing: every vertex $v \notin \mathcal{K}$ has a successor in \mathcal{K} , i.e., $\forall v \notin \mathcal{K}$, there is a vertex $k \in \mathcal{K}$ such that $(v, k) \in D$ [22].

A digraph possessing a kernel is called solvable [17].

Now, we associate (2) with Definition 2.1. Each Q_{r+1} in (2) is calculated by annihilating an element of strictly lower triangular part of A_r .

Let the indices of the elements annihilated after performing r steps be

$$(j_1, i_1), (j_2, i_2), \dots, (j_r, i_r). \quad (3)$$

Here, each step of (2) makes an element of strictly lower triangular part of A_r zero, and computation to do this can destroy pairs of zeros already created. So, let consider any distinct points

$$P_{(j_1, i_1)}, P_{(j_2, i_2)}, \dots, P_{(j_l, i_l)} \quad (l \leq r) \quad (4)$$

in the plane, which we shall call vertex. For every sequential annihilation, we connect the vertex $P_{(j_m, i_m)}$ to the vertex $P_{(j_{m+1}, i_{m+1})}$ by means of a directed edge $P_{(j_m, i_m)} P_{(j_{m+1}, i_{m+1})}$ direct from $P_{(j_m, i_m)}$ to $P_{(j_{m+1}, i_{m+1})}$. In this way, the annihilation process can be associated with a finite directed graph $G(A_{r+1}) = D(E, V)$.

To reduce a $(n \times n)$ nonsymmetric matrix by means of solvable finite directed graphs the elements to be annihilated and orthogonal transformation matrices are chosen strategically at each cycle of Stage1 and Stage2 of the D-NJLA.

The key problem is how to choose the elements to be eliminated and orthogonal matrices during the computation. These are the main topic that we discuss in the next section.

III. COMPUTATION OF EIGENVALUES OF A SQUARE NONSYMMETRIC MATRIX BY D-NJLA ALGORITHM

In this section, a Jacobi-like algorithm (which we call D-NJLA) is proposed. Before depict D-NJLA, we first give a necessary definition of matrices.

Definition 3.1: Let a non-symmetric $n \times n$ matrix A be divided into two blocks as follows:

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \quad (5)$$

where, the size of block $A_{1,1}, A_{2,2}, A_{1,2}$ and $A_{2,1}$ is $p \times p$, $n - p \times n - p$, $p \times n - p$ and $n - p \times p$, respectively.

If $\|A_{2,1}\|^2 = O(\epsilon)$, then A will be

$$\begin{bmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{bmatrix} \quad (6)$$

nearly 2×2 block upper triangular.

If $A_{1,1} = U_{1,1}$ and $A_{2,2} = U_{2,2}$ are upper triangular matrices, then A will be

$$\begin{bmatrix} U_{1,1} & A_{1,2} \\ A_{2,1} & U_{2,2} \end{bmatrix} \quad (7)$$

2×2 partitioned form [16].

In D-NJLA, at each cycle consist of two stages: Stage1 and Stage2.

Let $A = (a_{i,j}) = A_0^1$ be a nonsymmetric real $n \times n$ matrix.

In Stage1, a sequence of matrices $A_{r-}^k = (a_{i,j}^{r-,k})$, is obtained by a convergent sequence of successive transformations of the form :

$$A_{r-+1}^k = (Q_{r-+1}^k)^T A_r^k Q_{r-+1}^k \quad (A_0^k = \bar{A}_{r-+1}^{k-1}, \quad k \geq 2), \quad (8)$$

where $k \geq 1$ is the cycle number, $r^- \geq 0$ is the step number in k -th cycle and Q_{r-+1}^k is an orthogonal matrix in k -th cycle such that

$$Q_{r-+1}^k = \begin{bmatrix} 1 & & & & & \\ & c & & & -s & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ s & & & & & c & \\ & & & & & & 1 \end{bmatrix}, \quad (9)$$

where $c = \frac{y+1}{2}$ and $s = \frac{\sqrt{3-2y-y^2}}{2}$, with

$$0 \leq y \leq 1 \quad (10)$$

and

$$-3 \leq y \leq -2. \quad (11)$$

(10) and (11) correspond to $[0, \frac{\pi}{3}]$ and $[\frac{2\pi}{3}, \pi]$, respectively.

Case I: for $\Delta_{r-}^k > 0$, where Δ_{r-}^k is the discriminant of (51), yields

$$\begin{aligned} y_1 &= -1 + \sqrt{\frac{2\delta_{r-}^k}{K_{r-}^k} + \frac{\sqrt{\Delta_{r-}^k}}{2}} \\ y_2 &= -1 - \sqrt{\frac{2\delta_{r-}^k}{K_{r-}^k} + \frac{\sqrt{\Delta_{r-}^k}}{2}} \\ y_3 &= -1 + \sqrt{\frac{2\delta_{r-}^k}{K_{r-}^k} - \frac{\sqrt{\Delta_{r-}^k}}{2}} \\ y_4 &= -1 - \sqrt{\frac{2\delta_{r-}^k}{K_{r-}^k} - \frac{\sqrt{\Delta_{r-}^k}}{2}} \end{aligned} \quad (12)$$

and

Case II: for $\Delta_{r-}^k = 0$, yields

$$y_1 = -1 + \sqrt{\frac{2\delta_{r-}^k}{K_{r-}^k}} \quad y_2 = -1 - \sqrt{\frac{2\delta_{r-}^k}{K_{r-}^k}}, \quad (13)$$

where K_{r-}^k , L_{r-}^k and δ_{r-}^k are given by $(\|A_{r-}^k[i, j]\|_F^2 - 2D_{r-}^k(i, j)) = K_{r-}^k$, $(\xi_{r-}^k)^2 - (\xi_{r-}^k)^2 = L_{r-}^k$ and

$$K_{r-}^k - L_{r-}^k = \delta_{r-}.$$

Case III: for $\Delta_{r-}^k < 0$, the roots are out of the scope in our modification.

It is detailed in Appendix A and Appendix B how the orthogonal transformation matrix Q_{r-+1}^k and y roots are obtained.

At the k -th cycle of Stage1, after performing r^- steps we obtain A_{r-+1}^k , ($A_{0-}^k = \bar{A}_{r-+1}^{k-1}$) which is obtained at the $(k-1)$ -th cycle after performing r^+ steps and is in form like (6).

On the other hand, in the Stage2 a sequence of matrices $\bar{A}_{r+}^k = (\bar{a}_{i,j}^{r+,k})$, is obtained by a convergent sequence of successive transformations of the form :

$$\bar{A}_{r+1}^k = (Q_{r+1}^k)^T \bar{A}_{r+}^k Q_{r+1}^k \quad (\bar{A}_0^k = A_{r-+1}^k), \quad (14)$$

where $k \geq 1$ is the cycle number, $r^+ \geq 0$ is the step number in k -th cycle and Q_{r+1}^k is an orthogonal matrix given by (9) with (10) and (11). y is computed by (12) and (13), where K_{r+}^k , L_{r+}^k and δ_{r+} are given by $(\|A_{r+}^k[i,j]\|_F^2 - 2D_{r+}^k(i,j)) = K_{r+}^k$, $(\zeta_{r+}^k)^2 - (\xi_{r+}^k)^2 = L_{r+}^k$ and $K_{r+}^k - L_{r+}^k = \delta_{r+}$. At the k -th cycle of Stage2, after performing r^+ steps we obtain \bar{A}_{r+1}^k ($\bar{A}_{0+}^k = A_{r-+1}^k$), which is in form like (7).

IV. CHOOSING THE ELEMENT TO BE ANNIHILATED AND PROPER Q_{r+1}^k MATRICES DURING THE COMPUTATION

The characteristic equation of a $n \times n$ matrix $A = (a_{i,j})$ is, in general, an equation of the n th degree polynomial as given

$$\det(\mu I - A) = \mu^n + c_1 \mu^{n-1} + c_2 \mu^{n-2} + \dots + c_{n-1} \mu + c_n = 0. \quad (15)$$

If $\mu_1, \mu_2, \dots, \mu_n$ are eigenvalues of A , we have

$$\text{Tr}(A) = \sum_{i=1}^n a_{i,i} = \mu_1 + \mu_2 + \dots + \mu_n, \quad (16)$$

where $\text{Tr}(A)$ is the trace of A , and

$$\mathcal{D} = \mu_1 \times \mu_2 \times \dots \times \mu_n, \quad (17)$$

where \mathcal{D} is the determinant of A . Let $S_1 = \text{Tr}(A)$, $S_2 = \text{Tr}(A)^2, \dots, S_n = \text{Tr}(A)^n$. Then the coefficients c_1, c_2, \dots, c_n of the characteristic equation are

$$\begin{aligned} c_1 &= -S_1 \\ c_2 &= -\frac{1}{2}(c_1 S_1 + S_2) \\ c_3 &= -\frac{1}{3}(c_2 S_1 + c_1 S_2 + S_3) \\ &\dots \end{aligned} \quad (18)$$

$$c_n = -\frac{1}{n}(c_{n-1} S_1 + c_{n-2} S_2 + \dots + c_1 S_{n-1} + S_n).$$

Determinant of A can be calculated by

$$\mathcal{D} = (-1)^n c_n, \quad (19)$$

where c_n is given above [15].

From (19), the sign of \mathcal{D} , which is the product of eigenvalues of A , is related to the coefficients of the

characteristic equation. Therefore, we use the sign of the subdeterminant to choose the elements to be annihilated in the Stage1 and Stage2. So, before explaining the strategy of choosing the element to be annihilated, we give the following definition.

Definition 4.1: Let A is an $(n \times n)$ non-symmetric matrix, \mathcal{D} is the determinant of the matrix A and $S = ((1, 2), (1, 3), \dots, (1, n), (2, 3), (2, 4), \dots, (2, n), \dots, (n-1, n))$ be a finite sequence. Then a 2×2 submatrix of A is a matrix obtained from A by retaining only the elements belonging to rows with suffixes i, j columns with same suffixes i, j where $(i, j) \in S$. A 2×2 submatrix of A will be denoted by

$$A[i, j] = \begin{bmatrix} a_{i,i} & a_{i,j} \\ a_{j,i} & a_{j,j} \end{bmatrix} \quad (20)$$

Also, determinant of the submatrix $A[i, j]$ is 2×2 subdeterminant that will be denoted by

$$\mathcal{D}(i, j) = \begin{vmatrix} a_{i,i} & a_{i,j} \\ a_{j,i} & a_{j,j} \end{vmatrix} = a_{i,i} a_{j,j} - a_{i,j} a_{j,i} \quad (21)$$

where $((i, j) \in S)$ [16].

(21) is $\mathcal{D}(i, j) \geq 0$ or $\mathcal{D}(i, j) < 0$.

The annihilation strategy and choosing proper orthogonal transformation matrices of Stage1 and Stage2 is as follows with the help of Definition 4.1.

In the r^- -th step of the k -th cycle of Stage1, while annihilating the strict lower triangular elements A_{r-}^k , to every r^-, k there corresponds a pair (j^*, i^*) of two indices i^* and j^* (satisfying $1 \leq i^* < j^* \leq n$) which are indices of strictly lower triangular part element largest in magnitude and satisfy $\mathcal{D}(i^*, j^*) < 0$

$$|a_{j^*, i^*}^{r-, k}| = \{\max |a_{j, i}^{r-, k}| \mid \wedge \mathcal{D}(i, j) < 0, \quad i < j \leq n\}. \quad (22)$$

From (12) and (13), there are at least two and at most four Q_{r-+1}^k matrices that will be used to annihilate $a_{j^*, i^*}^{r-, k}$. Also, for Q_{r-+1}^k matrices there are $0 \leq y \leq 1$ and $-3 \leq y \leq -2$ intervals (given by (10) and (11)). So, in order to annihilate $a_{j^*, i^*}^{r-, k}$, one must choose from among the Q_{r-+1}^k matrices.

The procedure used to find the proper Q_{r-+1}^k matrices during the Stage1 computations is given as follows:

- 1) If $\delta_{r-}^k > 0$
 - a) If Δ_{r-}^k is positive or zero then calculate Q_{r-+1}^k transformation matrices by using (12) or (13) roots.
 - b) Exclude both extraneous transformation matrices and the transformation matrices which correspond to the roots not fall into $0 \leq y \leq 1$ or $-3 \leq y \leq -2$ interval.
 - c) If the number of remaining roots is one then, the transformation matrix which corresponds to this root is used for annihilation.
 - d) If the number of remaining roots is two and fall into same interval ($0 \leq y \leq 1$ or $-3 \leq y \leq -2$)

then, choose the transformation matrix that is the closest identity.

- e) If the number of remaining roots is two and fall into different interval then, choose the transformation matrix corresponding to the root falling in $-3 \leq y \leq -2$ interval.
- f) If $\Delta_{r-}^k < 0$ then, skip without annihilation in the next steps because the roots are out of the scope.

- 2) If $\delta_{r-}^k < 0$ then, skip without annihilation in the next steps because the roots are complex numbers.

On the other hand, in the r^+ -th step of the k -th cycle of Stage2, while annihilating the strict lower triangular elements of \bar{A}_{r+}^k , to every r^+, k there corresponds a pair (j^{**}, i^{**}) of two indices i^{**} and j^{**} (satisfying $1 \leq i^{**} < j^{**} \leq n$) which are indices of strictly lower triangular part element largest in magnitude and satisfy $\mathcal{D}(i^{**}, j^{**}) \geq 0$

$$|\bar{a}_{j^{**}, i^{**}}^{r^+, k}| = \{\max |\bar{a}_{j, i}^{r^+, k}| \wedge \mathcal{D}(i, j) \geq 0, i < j \leq n\}. \quad (23)$$

Again we can say here, from (12) and (13), there are at least two and at most four $Q_{r^++1}^k$ matrices that will be used to annihilate $\bar{a}_{j^{**}, i^{**}}^{r^+, k}$. Also, for the $Q_{r^++1}^k$ matrices there are $0 \leq y \leq 1$ and $-3 \leq y \leq -2$ intervals (given by (10) and (11)). So, in order to annihilate $\bar{a}_{j^{**}, i^{**}}^{r^+, k}$, one must choose from among the $Q_{r^++1}^k$ matrices.

The procedure used to find the proper $Q_{r^++1}^k$ matrices during the Stage2 computations is the same except that

- If the number of remaining roots is two and fall into different interval then, choose the transformation matrix corresponding to the root falling in $0 \leq y \leq 1$.

item is used instead of (1e)-th item and $(\|\bar{A}_{r+}^k[i, j]\|_F^2 - 2D_{r+}^k(i, j)) = K_{r+}^k, (\zeta_{r+}^k)^2 - (\xi_{r+}^k)^2 = L_{r+}^k, K_{r+}^k - L_{r+}^k = \delta_{r+}.$

Corollary 1: At the end of each of the Stage1 and Stage2 of the D-NJLA, the matrix forms obtained are similar to (5) and (7), respectively, since the eigenvalues are not arranged with the same signs consecutively on the diagonal. Stage1 and Stage2 continue sequentially until the matrix A is reduced to the upper triangular form.

Corollary 2: Each of the Stage1 and Stage2 of the D-NJLA produce zero blocks and a solvable directed graph whose vertices are indices of elements to be annihilated, and consecutive computation of Stage1 and Stage2 can destroy zero blocks already created. But, as the iteration progress of Stage1 and Stage2, $off(A_{r-}^k)$ and $off(\bar{A}_{r+}^k)$ decrease, leaving approximations to the eigenvalues on the diagonal, where $off(A_{r-}^k) = \sqrt{\sum_{j>i}^n (a_{ji}^{r-, k})^2}$ and $off(\bar{A}_{r+}^k) = \sqrt{\sum_{j>i}^n (\bar{a}_{ji}^{r+, k})^2}$.

The basic structure of D-NJLA is sketched in Algorithm IV. In this algorithm the value of k is equal to the number of cycles and, the values of H and H_1 show the indices of the elements that were skipped without being annihilated in the k -th cycle of the Stage1 and Stage2, respectively. The process stops if the value of offnorm which equals to $off(\bar{A}_{r+}^k)$ for

$k \geq 2$ is constant.

Algorithm 1. Finding eigenvalues of a real nonsymmetric $n \times n$ matrix with D-NJLA

input : A $n \times n$ real nonsymmetric matrix A

output: An upper triangular matrix whose diagonal elements are eigenvalues of the matrix A

Begin

$\epsilon = eps,$

$k = 0$

$offnorm(0) = 0,$

$offnorm(1) = \sqrt{\sum_{i<j \leq n} a_{i,j}^2}$

while $offnorm(k) \neq offnorm(k+1)$ **do**

$k = k + 1, \quad r^-, r^+ = 0, \quad H, H_1 \leftarrow \emptyset$

if $k > 1$ **then**

$A_0^k = \bar{A}_{r^++1}^k$

$maxoff(A_0^k) = 1$ // initial value

while $maxoff(A_{r-}^k) = \max_{j>i} |a_{j,i}^{r-, k}| > \epsilon$ **do**

$r^- = r^- + 1$

$M \leftarrow \emptyset$

Find (j^*, i^*) according to the (22) such that

$(j^*, i^*) \notin H$

$\xi_{r-}^k = a_{i^*, j^*}^{r-, k},$

$\zeta_{r-}^k = a_{j^*, i^*}^{r-, k},$

$\varrho_{r-}^k = a_{j^*, j^*}^{r-, k} - a_{i^*, i^*}^{r-, k},$

$K_{r-}^k = (\|A_{r-}^k[i, j]\|_F^2 - 2D_{r-}^k(i, j)),$

$L_{r-}^k = (\zeta_{r-}^k)^2 - (\xi_{r-}^k)^2,$

$\delta_{r-}^k = K_{r-}^k - L_{r-}^k$

$\Delta_{r-}^k = \frac{16}{(K_{r-}^k)^2} ((\delta_{r-}^k)^2 - 4(\xi_{r-}^k)^2)$

if $\delta_{r-}^k > 0$ **and** $(\Delta_{r-}^k > 0$ **or** $\Delta_{r-}^k = 0)$ **then**

compute Q_{r-+1}^k transformations by (12) or (13) and find M by extracting both extraneous transformations and those corresponding to y roots not fall into (10) or (11) among them

if $|M| = 1$ **then**

use Q_{r-+1}^k which corresponds to this root

if $|M| = 2$ **then**

if both roots fall into (10) or (11) then

use Q_{r-+1}^k which is close to identity matrix

if one root falls into (10) and the other falls into (11) then

use Q_{r-+1}^k which corresponds y root falls into (11)

if $(|M| = 0$ **or** $\delta_{r-}^k < 0)$ **or** $\Delta_{r-}^k < 0$ **then**

skip without annihilation

$H \leftarrow H \cup \{(j^*, i^*)\}$

$A_{r-+1}^k = (Q_{r-+1}^k)^T A_{r-}^k Q_{r-+1}^k$

consider all $a_{i,j}^{r-, k}$ ($i \neq j$) as zero if $|a_{i,j}^{r-, k}| < eps$

$off(A_{r-+1}^k)$

end

end

performing r^- step of k -th cycle of the Stage1. Additionally, we denote

$$\begin{aligned}
 \xi_{r^-}^k &= \max_{i,j=1..n} |a_{j,i}^{r^-,k}| \\
 \zeta_{r^-}^k &= \max_{i < j \leq n} |a_{j,i}^{r^-,k}| \\
 \varrho_{r^-}^k &= \min_{i \neq j} |a_{j,j}^{r^-,k} - a_{i,i}^{r^-,k}| \\
 K_{r^-}^k &= \min_{i \neq j} |(a_{j,j}^{r^-,k} - a_{i,i}^{r^-,k})^2 + (a_{j,i}^{r^-,k} + a_{i,j}^{r^-,k})^2| \\
 L_{r^-}^k &= \max_{i \neq j} |(a_{j,i}^{r^-,k})^2 - (a_{i,j}^{r^-,k})^2| \\
 \delta_{r^-}^k &= (K_{r^-}^k - L_{r^-}^k)
 \end{aligned} \tag{24}$$

such that 2×2 subdeterminant is negative. Here, $\xi_{r^-}^k$ is the modulus of the largest element in modulus of $A_{r^-}^k$ such that 2×2 subdeterminant is negative. The modulus of the largest element in modulus of the strictly lower triangular part of $A_{r^-}^k$ is $\zeta_{r^-}^k$ such that 2×2 subdeterminant is negative. The smallest distance between to two diagonal elements of the matrix $A_{r^-}^k$ is $\varrho_{r^-}^k$ such that 2×2 subdeterminant is negative. And, the minimum difference between the $K_{r^-}^k$ and $L_{r^-}^k$ values of the matrix $A_{r^-}^k$ is $\delta_{r^-}^k$ such that 2×2 subdeterminant is negative. In the following, let r^- be fixed and suppose we have

$$\delta_{r^-}^k > 0 \quad \text{and} \quad 4 \frac{K_{r^-}^k (\zeta_{r^-}^k)^2}{(\delta_{r^-}^k)^2} < 1. \tag{25}$$

Assume that the (j^*, i^*) -element of $A_{r^-}^k$ is the pivot element of the current $((r^- + 1)$ st) step of k -th cycle of Stage1. We compute the orthogonal matrix

$$Q = \begin{bmatrix} \frac{y+1}{2} & -\frac{\sqrt{3-2y-y^2}}{2} \\ \frac{\sqrt{3-2y-y^2}}{2} & \frac{y+1}{2} \end{bmatrix} \tag{26}$$

that satisfies

$$Q^T \begin{bmatrix} a_{i^*i^*}^{r^-,k} & a_{i^*j^*}^{r^-,k} \\ a_{j^*i^*}^{r^-,k} & a_{j^*j^*}^{r^-,k} \end{bmatrix} Q = \begin{bmatrix} a_{i^*i^*}^{r^-+1,k} & a_{i^*j^*}^{r^-+1,k} \\ 0 & a_{j^*j^*}^{r^-+1,k} \end{bmatrix}, \tag{27}$$

that is as close as possible to identity among all matrices satisfying (27) and which corresponds to the rotation matrix whose rotation angle falls into $[-3, -2]$ as much as possible.

To obtain an upper bound for the modulus of $\frac{\sqrt{3-2y-y^2}}{2}$, we will use the following lemma, that is similar to Theorem V.2.1 in [11].

Lemma 1: Let $A \in \mathbb{R}$ such that

$$A = \begin{bmatrix} a & \xi \\ \zeta & b \end{bmatrix}, \tag{28}$$

where $4 \frac{K\zeta^2}{\delta^2} < 1$, $K = \|A[1, 2]\|_F^2 - 2\mathcal{D}(1, 2)$, $L = \zeta^2 - \xi^2$, $(K - L) = \delta > 0$ and $\mathcal{D}(1, 2) < 0$. Then there exist a unique eigenvector ν of A satisfying

$$\nu = \begin{bmatrix} 1 \\ p \end{bmatrix} \quad \text{and} \quad |p| < \frac{\sqrt{2}\zeta}{\sqrt{\delta}}. \tag{29}$$

(3)

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 $\bar{A}_0^k = A_{r^-+1}^k$ 
 $\text{maxoff}(\bar{A}_{r^+}^k) = 1$ 
while  $\text{maxoff}(\bar{A}_{r^+}^k) = \max_{j>i} |\bar{a}_{j,i}^{r^+,k}| > \epsilon$  do
     $r^+ = r^+ + 1$ ,
     $M \leftarrow \emptyset$ 
    Find  $(j^{**}, i^{**})$  according to the (22) such that
     $(j^{**}, i^{**}) \notin H_1$ 
     $\xi_{r^+}^k = \bar{a}_{i^{**},j^{**}}^{r^+,k}$ ,
     $\zeta_{r^+}^k = \bar{a}_{j^{**},i^{**}}^{r^+,k}$ ,
     $\varrho_{r^+}^k = \bar{a}_{j^{**},j^{**}}^{r^+,k} - \bar{a}_{i^{**},i^{**}}^{r^+,k}$ ,
     $K_{r^+}^k = (\|\bar{A}_{r^+}^k[i, j]\|_F^2 - 2D_{r^+}^k(i, j))$ ,
     $L_{r^+}^k = (\zeta_{r^+}^k)^2 - (\xi_{r^+}^k)^2$ ,  $\delta_{r^+} = K_{r^+}^k - L_{r^+}^k$ 
     $\Delta_{r^+}^k = \frac{16}{(K_{r^+}^k)^2} ((\delta_{r^+}^k)^2 - 4(\xi_{r^+}^k)^2)$ 
    if  $\delta_{r^+}^k > 0$  and  $(\Delta_{r^+}^k > 0 \text{ or } \Delta_{r^+}^k = 0)$  then
        compute  $Q_{r^++1}^k$  transformations by (12) or
        (13) and find  $M$  by extracting both extraneous
        transformations and those corresponding to  $y$ 
        roots not fall into (10) or (11) among them
        if  $|M| = 1$  then
            use  $Q_{r^++1}^k$  which corresponds to this root
        if  $|M| = 2$  then
            if both roots fall into (10) or (11) then
                use  $Q_{r^++1}^k$  which corresponds to be
                close to identity
            if one root falls into (10) and the other falls
            into (11) then
                use  $Q_{r^++1}^k$  which corresponds  $y$  root
                falls into (10)
        if  $(|M| = 0 \text{ or } \delta_{r^+}^k < 0) \text{ or } \Delta_{r^+}^k < 0$  then
            skip without annihilation
             $H_1 \leftarrow H \cup \{(j^{**}, i^{**})\}$ 
             $\bar{A}_{r^++1}^k = (Q_{r^++1}^k)^T \bar{A}_{r^+}^k Q_{r^++1}^k$ 
            consider all  $\bar{a}_{i,j}^{r^+,k}$  ( $i \neq j$ ) as zero if  $|\bar{a}_{i,j}^{r^+,k}| < \epsilon$ 
             $\text{off}(\bar{A}_{r^++1}^k)$ 
     $\text{offnorm}(k+1) = \text{off}(\bar{A}_{r^++1}^k)$ 

```

V. ASYMPTOTIC QUADRATIC CONVERGENCE OF THE D-NJLA

The proof of asymptotic quadratic convergence of the classic symmetric Jacobi method uses monotonicity of the offnorm, i.e., offnorm converges to a limit [14], [23], [24]. For the nonsymmetric case, the offnorm is no longer monotone. So, the possible changes in the moduli of entries in the strictly lower triangular part of the matrix in a single step of k -th cycle of Stage1 will be investigated in detail. Let $A_{0^-}^k$ be the starting matrix, $A_{r^-}^k = (a_{ij}^{r^-,k})$ be the matrix obtained after

Using this lemma and (25), we find that $|\frac{\sqrt{3-2y-y^2}}{2}| < \frac{\sqrt{2}\zeta_{r-}^k}{\sqrt{\delta_{r-}^k}}$.

Using the above lemma and the following theorem, we will show that k -th cycle of the Stage1 is quadratically convergent.

Theorem 1: Let $A = (a_{ij}) = A_{0-}^1 \in \mathbb{R}^{n \times n}$, and let $A_{r-}^k = (a_{ij}^{r-,k})$ denote the matrix that is obtained from $\bar{A}_{r+}^{k-1} = A_{0-}^k$, $k \geq 2$, (if $k = 1$, $A_{0-}^0 = A$) after performing $r-$ Stage1 steps of D-NJLA. Let ζ_{0-}^k and $\zeta_{\mu-}^k$ be the largest modulus of an element in the strictly lower triangular part of A_{0-}^k and $A_{\mu-}^k$ that is obtained from A_{0-}^k after having completed $\mu-$ sweeps such that 2×2 subdeterminant is negative, i.e.,

$$\psi_{\mu-}^k = \max_{i < j \leq n} |a_{ji}^{\mu-,N,k}|, \mu- \in \mathbb{N} \cup \{0\} \quad (30)$$

and let ϕ_{0-}^k and $\phi_{\mu-}^k$ be the largest modulus of an element in $A_{0-}^k = \bar{A}_{r+}^{k-1}$ and $A_{\mu-}^k$ i.e.,

$$\phi_{\mu-}^k = \max_{i,j \leq n} |a_{ij}^{\mu-,N,k}|, \mu- \in \mathbb{N} \cup \{0\} \quad (31)$$

such that 2×2 subdeterminant is negative. Moreover, set $N = n_1 n_2$, where n_1 and n_2 are the number of negative and positive signed eigenvalues of A , respectively and

$$\begin{aligned} \nu_-^k &= \|A\|_F, \quad \varrho_-^k = \frac{1}{2} \min |a_{ii} - a_{jj}|, (i \neq j), \\ K_-^k &= \min_{i \neq j} |(a_{ii} - a_{jj})^2 + (a_{ij} + a_{ji})^2|, \\ L_-^k &= \max_{i \neq j} |a_{ji}^2 - a_{ij}^2|, \quad \delta_-^k = \frac{1}{2}(K_-^k - L_-^k), \\ \phi_-^k &= 2\psi_{0-}^k \left(1 + \frac{\sqrt{2}\nu_-^k}{\sqrt{\delta_-^k}}\right)^{N+1}, \\ \psi_-^k &= 2\psi_{0-}^k \left(1 + \frac{\sqrt{2}\nu_-^k}{\sqrt{\delta_-^k}}\right)^N. \end{aligned} \quad (32)$$

If $\delta_-^k > 0$ and if ψ_{0-}^k is sufficiently small such that

$$\begin{aligned} \frac{K_-^k (\psi_-^k)^2}{(\delta_-^k)^2} &< \frac{1}{4}, \\ \left(\frac{\sqrt{2}(\psi_-^k)^2}{\sqrt{\delta_-^k}} + \nu_-^k \frac{4(\psi_-^k)^2}{\delta_-^k} + \nu_-^k \frac{\sqrt{2}\psi_-^k}{\sqrt{\delta_-^k}} \right) &\leq \frac{\varrho_-^k}{2N}, \\ \left(2\nu_-^k \left(\frac{\sqrt{2}(\psi_-^k)^2}{\sqrt{\delta_-^k}} + \nu_-^k \frac{4(\psi_-^k)^2}{\delta_-^k} + \nu_-^k \frac{\sqrt{2}\psi_-^k}{\sqrt{\delta_-^k}} \right) - \right. \\ &\quad \left. \left(\frac{\sqrt{2}(\psi_-^k)^2}{\sqrt{\delta_-^k}} + \nu_-^k \frac{4(\psi_-^k)^2}{\delta_-^k} + \nu_-^k \frac{\sqrt{2}\psi_-^k}{\sqrt{\delta_-^k}} \right)^2 - \right. \\ &\quad \left. \frac{2(\psi_-^k)^4}{\delta_-^k} + \nu_-^k \psi_-^k \right) &\leq \frac{\delta_{0-}^k}{4N}, \\ \frac{\sqrt{2}N}{\sqrt{\delta_-^k}} \left(1 + \frac{\sqrt{2}\nu_-^k}{\sqrt{\delta_-^k}} \right) (\psi_-^k)^2 &\leq \psi_{0-}^k \end{aligned} \quad (33)$$

then for all $\mu- \in \mathbb{N} \cup \{0\}$, we have that

$$\psi_{(\mu-+1)N}^k \leq N \frac{\sqrt{2}}{\sqrt{\delta_-^k}} \left(1 + \frac{\sqrt{2}\nu_-^k}{\sqrt{\delta_-^k}} \right)^{2N+1} (\psi_{\mu-}^k)^2, \quad (34)$$

i.e., k -th cycle of the Stage1 of D-NJLA sweep converges quadratically over the number of sweeps.

Similarly, it can be said k -th cycle of the Stage2 of D-NJLA sweep converges quadratically over the number of sweeps. So, we can say cycles of the Stage1 and Stage2 converge quadratically, i.e., the D-NJLA converges quadratically.

a) Complexity of D-NJLA: D-NJLA calculates the eigenvalues of nonsymmetric matrices with solvable directed graphs. At each step of the Stage 1 of the D-NJLA, an entry, which has the property that the strictly lower triangular part element has the largest magnitude and the sign of 2×2 subdeterminant is negative, is chosen. At each step of the Stage2, an entry, which has the property that the strictly lower triangular part element has the largest magnitude and the sign of 2×2 subdeterminant is zero or positive, is chosen. Therefore, $\frac{n(n-1)}{2}$ elements must be compared in each step of both Stage1 and Stage2. So, D-NJLA's complexity is $O(n^2)$.

VI. EXPERIMENTAL RESULTS

We implemented the D-NJLA in MATLAB and run it on an Intel Core i7 processor (2.7 GHz) computer. An off-diagonal element $a_{i,j}$ is considered zero if $|a_{i,j}| < \text{eps}$, where eps is a machine dependent constant. As a stopping criteria we used $\text{off}(A) = \sqrt{\sum_{i < j \leq n} a_{i,j}^2} = \text{constant}$.

We illustrate the D-NJLA with the following numerical examples. We give the "effective" number of sweeps of the Stage 1 in k -th cycle, that is the number of rotations divided by $n_1 n_2$. "Effective" number of sweeps of the Stage 2 in k -th cycle is the number of rotations divided by $\frac{n_1(n_1-1)}{2} + \frac{n_2(n_2-1)}{2}$. Here, n_1 and n_2 are the number of negative and positive signed eigenvalues of the given matrix, respectively.

We take matrices from [2], which are 7×7 'Grund/b1_ss', 9×9 'vanHeukelum/cage4', 102×102 'MathWorks/pivtol', 135×135 'Rajat/rajat11' and 216×216 'Fidap/ex1' and reduce them to upper triangular form.

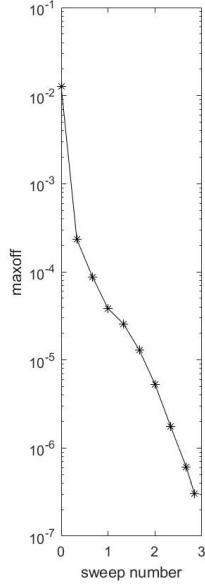


Fig. 1. Convergence behaviour of the Stage 2 of the D-NJLA for 102×102 'MathWorks/pivtol'

Fig. 1 displays the typical convergence behaviour of the Stage1 of the D-NJLA for 102×102 'MathWorks/pivtol'. Since all eigenvalues of 'MathWorks/pivtol' are positive real numbers, 'MathWorks/pivtol' was reduced in the Stage2 of the D-NJLA.

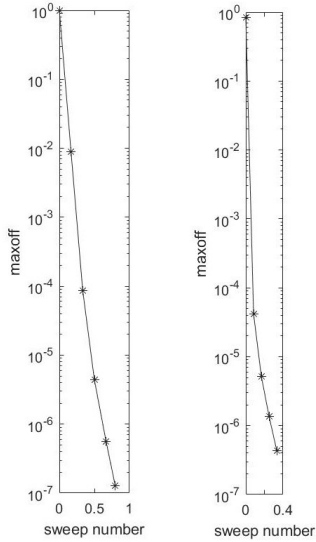


Fig. 2. Convergence behaviour of the Stage 1 and Stage 2 of the D-NJLA for 135×135 'Rajat/rajat11'

Fig. 2 displays the typical convergence behaviour of the Stage1 and Stage2 of the D-NJLA for 135×135 'Rajat/rajat11'. Since 'Rajat/rajat11' have both positive and negative real eigenvalues, 'Rajat/rajat11' was reduced in both Stage1 and Stage2 of the D-NJLA.

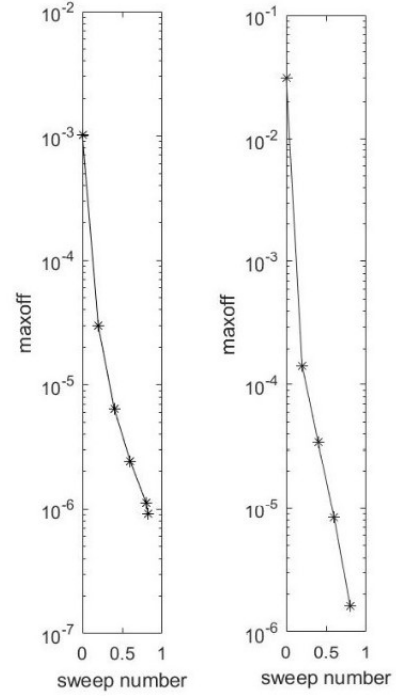


Fig. 3. Convergence behaviour of the Stage 1 and Stage 2 of the D-NJLA for 216×216 'Fidap/ex1'

Finally, we illustrate the typical convergence behaviour of the Stage1 and Stage2 of the D-NJLA for 216×216 'Fidap/ex1', see Fig. 3. 216×216 'Fidap/ex1', which have both positive and negative real eigenvalues, was reduced in both Stage1 and Stage2 of the D-NJLA.

The matrices given as an example above have a large number of very close eigenvalues. Therefore, as the number of steps increases in Stage2, ϱ_{r+} increases (i.e., δ_{r+}^k decreases) considerably, so $N \frac{\sqrt{2}}{\sqrt{\delta_+^k}} \left(1 + \frac{\sqrt{2}\nu_+^k}{\sqrt{\delta_+^k}} \right)^{2N+1}$ increases considerably. Thus, the rate of convergence is expected to slow down significantly. But, as can be seen from Fig. 2 and Fig. 3, no significant slowdown in the convergence rate in the Stage2 was observed.

In all figures, no stationary phase was observed in both the Stage1 and Stage2. As expected, it can be observed $\max\text{off}(A)$ decreases over the number of sweeps of the Stage1 and Stage2. As predicted by the theory, the convergence rate of the Stage1 and Stage2 becomes asymptotically quadratic.

Also, we take matrices, some of whose eigenvalues are complex and whose the number of eigenvectors is less than the order of the taken matrix. These matrices are reduced nearly upper triangular form by D-NJLA and the algorithm converges linearly for this type of matrices because the roots have not been taken into consideration for $\Delta_{r-}^k < 0$ and $\Delta_{r+}^k < 0$. Additionally, the matrices, which have a zero eigenvalue are reduced upper triangular form by the D-NJLA and the convergence behaviour of the D-NJLA is asymptotic

quadratic and no stationary phase is observed.

VII. CONCLUSIONS

In this paper, we have presented the details of D-NJLA, which is a Jacobi-like algorithm. Moreover, we also present numerical experiments with the D-NJLA, that is applied to some examples of real nonsymmetric matrices to reduce them to upper triangular matrices. As seen the given examples, the convergence behavior of the D-NJLA is asymptotic quadratic and no stationary phase is observed, although the matrices given as an example have a large number of very close eigenvalues.

Still, while we encouraged that the D-NJLA works as it does, there are many questions unanswered. Firstly, a proof of global convergence is still not provided. The D-NJLA is not adapted for matrices with complex eigenvalues. Therefore, the cases of $\Delta_{r-}^k < 0$ and $\Delta_{r+}^k < 0$ or the equivalences of (37) and (38) should be taken into consideration. So, the algorithm can be adapted for this type matrices. Also, considering the matching between the independence set of graphs and groups [1], a group theory approach can be made to the algorithm.

APPENDIX A

ORTHOGONAL TRANSFORMATION MATRIX

The corresponding $(n \times n)$ orthogonal transformation matrix Q_r^k , analogous to the rotation of an axis in the (j, i) plane may be given by:

$$R_r^k = \begin{bmatrix} 1 & & & & \\ & \cos \theta & & & -\sin \theta \\ & & 1 & & \\ & & & \ddots & \\ & \sin \theta & & & 1 & \cos \theta \\ & & & & & & 1 \end{bmatrix} \quad (35)$$

which corresponds to the transformation matrix of the Jacobi method [3]. For more details on the variance of the transformation matrix see [21].

When A is a nonsymmetric matrix, the angle between some eigenvectors can be quite small. In this case, as stated above, the numerical computation of the angle θ , which depends on angles between eigenvectors, can be more delicate [8].

To overcome this problem, we expressed $\cos \theta$ and $\sin \theta$ with quadratic polynomials as explained below and the rotation parameter λ^2 corresponding to the rotation angle θ was taken in the following intervals.

The characteristic equation of (35) is given by:

$$\det(R_r - \lambda I) = (1 - \lambda)^{n-2}(\lambda^2 - 2\lambda \cos \theta + 1) = 0. \quad (36)$$

To compute the eigenvalues of (36), we set either $(1 - \lambda)^{n-2}$ or $(\lambda^2 - 2\lambda \cos \theta + 1)$. To avoid complex operations, we consider $(\lambda^2 - 2\lambda \cos \theta + 1)$, hence the rotation angle may be written in terms of λ parameter as:

$$\cos \theta = \frac{\lambda^2 + 1}{2\lambda}. \quad (37)$$

In this case, substituting (37) to $\sin \theta = \sqrt{1 - \cos^2 \theta}$, we obtain

$$\sin \theta = \pm \frac{|\lambda^2 - 1|}{2|\lambda|}i, \quad (38)$$

which is a complex number. To determine the real interval for θ , without taking λ in the denominator of (37) we use

$$\cos \theta = \frac{\lambda^2 + 1}{2}, \sin \theta = \frac{\sqrt{3 - 2\lambda^2 - \lambda^4}}{2}, \quad (39)$$

which determines rotation angle θ of transformation matrix in terms of the λ^2 parameter is obtained. Hence, for new transformation matrix, we have:

$$Q_r^k = \begin{bmatrix} 1 & & & & \\ & c & & & -s \\ & & 1 & & \\ & & & \ddots & \\ & s & & & 1 & c \\ & & & & & & 1 \end{bmatrix}, \quad (40)$$

where $y = \lambda^2$, $c = \frac{y+1}{2}$ and $s = \frac{\sqrt{3-2y-y^2}}{2}$. So, the angle θ is chosen for appropriate the y parameter, thus, we have

$$0 \leq y \leq 1 \quad (41)$$

and

$$-3 \leq y \leq -2. \quad (42)$$

For example, the interval $0 \leq y \leq 1$ corresponds to the interval $0 \leq \theta \leq \frac{\pi}{3}$ for the rotation angle θ , and the second interval corresponds to $\frac{2\pi}{3} \leq \theta \leq \pi$.

APPENDIX B

CALCULATION OF THE ROTATION PARAMETER y

Let $(a_{j,i}^{r,k})$ be any element to be annihilated in Stage1 or Stage2 such that r is any step number in k -th cycle.

The element $(a_{j,i}^{r,k})$ is annihilated by a new similarity transformation matrix Q_{r+1}^k . This annihilation affects the entries in rows and columns j and i only. The modified $a_{j,i}^{r+1,k}$ element is given by:

$$a_{j,i}^{r+1,k} = (a_{j,j}^{r,k} - a_{i,i}^{r,k}) \frac{y+1}{2} \frac{\sqrt{3-2y-y^2}}{2} + a_{j,i}^{r,k} \frac{(y+1)^2}{4} - a_{i,j}^{r,k} \frac{(3-2y-y^2)}{4}. \quad (43)$$

The y parameter is chosen to annihilate $a_{j,i}^{r+1,k}$ element. So, we obtain a corresponding equation for the y parameter

$$(a_{j,j}^{r,k} - a_{i,i}^{r,k}) \frac{(y+1)\sqrt{3-2y-y^2}}{4} + a_{j,i}^{r,k} \frac{(y+1)^2}{4} - a_{i,j}^{r,k} \frac{(3-2y-y^2)}{4} = 0 \quad (44)$$

for nonsymmetric matrices.

Hence, the transformation matrix Q_{r+1}^k will be determined in terms of the solutions for y of (44). Equation (44) can be solved as follows;

First, we rewrite (44) as:

$$\varrho_r^k(y+1)\sqrt{3-2y-y^2} = \xi_r^k(3-2y-y^2) - \zeta_r^k(y^2+2y+1), \quad (45)$$

where

$$(a_{j,j}^{r,k} - a_{i,i}^{r,k}) = \varrho_r^k, a_{i,j}^{r,k} = \xi_r^k, a_{j,i}^{r,k} = \zeta_r^k, \quad (46)$$

then by squaring (45) we have

$$\begin{aligned} & ((\varrho_r^k)^2 + (\xi_r^k)^2 + (\zeta_r^k)^2 + 2\xi_r^k\zeta_r^k)y^4 + 4((\varrho_r^k)^2 + (\xi_r^k)^2 + (\zeta_r^k)^2 + 2\eta_r^k\zeta_r^k)y^3 + (2(\varrho_r^k)^2 + 6(\xi_r^k)^2 - 2(\xi_r^k)^2 + 4\xi_r^k\zeta_r^k)y^2 + (-8\xi_r^k\zeta_r^k + 4(\xi_r^k)^2 - 12(\xi_r^k)^2 - 4(\varrho_r^k)^2)y + ((\xi_r^k)^2 - 6\xi_r^k\zeta_r^k + 9(\xi_r^k)^2 - 3\varrho_r^k)^2 = 0 \end{aligned} \quad (47)$$

and

$$\begin{aligned} & (\|A_r^k[i,j]\|_F^2 - 2\mathcal{D}_r^k(i,j))y^4 + 4(\|A_r^k[i,j]\|_F^2 - 2\mathcal{D}_r^k(i,j))y^3 + (2(\|A_r^k[i,j]\|_F^2 - 2\mathcal{D}_r^k(i,j)) + 4((\xi_r^k)^2 - (\xi_r^k)^2))y^2 + (8(\xi_r^k)^2 - (\xi_r^k)^2 - 4(\|A_r^k[i,j]\|_F^2 - 2\mathcal{D}_r^k(i,j))y + 4((\xi_r^k)^2 - (\xi_r^k)^2) - 3(\|A_r^k[i,j]\|_F^2 - 2\mathcal{D}_r^k(i,j)) + 16(\xi_r^k)^2 = 0, \end{aligned} \quad (48)$$

where

$$(\varrho_r^k)^2 + 2a_{i,i}^{r,k}a_{j,j}^{r,k} + (\xi_r^k)^2 + (\zeta_r^k)^2 = \|A_r^k[i,j]\|_F^2, \quad \varrho_r^k\xi_r^k - a_{i,i}^{r,k}a_{j,j}^{r,k} = \mathcal{D}_r^k(i,j) \quad (49)$$

and $\mathcal{D}_r^k(i,j)$ is defined by Definition 4.1.

Letting $y = z - 1$, (48) simplifies to

$$\begin{aligned} & z^4 + 4\left(\frac{(\xi_r^k)^2 - (\xi_r^k)^2}{(\|A_r^k[i,j]\|_F^2 - 2\mathcal{D}_r^k(i,j))} - 1\right)z^2 + 16\frac{(\xi_r^k)^2}{(\|A_r^k[i,j]\|_F^2 - 2\mathcal{D}_r^k(i,j))} = 0 \end{aligned} \quad (50)$$

and letting $z^2 = t$, (50) simplifies to

$$t^2 + 4\left(1 - \frac{L_r^k}{K_r^k}\right)t + 16\frac{(\xi_r^k)^2}{K_r^k} = 0, \quad (51)$$

where

$$\|A_r^k[i,j]\|_F^2 - 2\mathcal{D}_r^k(i,j) = K_r^k \quad \text{and} \quad (\xi_r^k)^2 - (\xi_r^k)^2 = L_r^k. \quad (52)$$

Solving (51) for t and using the change of variables in reverse the eigenvalues of (44), thus, we obtain Case I, Case II and Case III, where $\delta_r^k = K_r^k - L_r^k$ and $\Delta_r^k = \frac{16}{(K_r^k)^2}((\delta_r^k)^2 - 4(\xi_r^k)^2 K_r^k)$.

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