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RESEARCH ARTICLE

Solving Time-Space Fractional Boussinesq Equation Using Homotopy Perturbation Method

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ABSTRACT

This paper aims to implement the homotopy perturbation technique to solve the time-space fractional Boussinesq equation, a significant model in the analysis of nonlinear wave propagation. Through the application of the homotopy perturbation technique, we derive analytical expressions for the solutions of the time-space fractional Boussinesq equation and validate these solutions through comparisons with numerical methods. Obtain results demonstrate the efficiency and accuracy of the homotopy perturbation method in solving the time-space fractional Boussinesq equation.

Keywords: Fractional calculus, homotopy perturbation method, nonlinear wave propagation, time-space fractional Boussinesq equation.

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1. Introduction

The study of fractional differential equations has gained significant attention in recent years due to their ability to model complex phenomena exhibiting memory and hereditary properties across various scientific disciplines, including physics [1]–[3], engineering [4], biology [5], and finance. Fractional calculus extends the traditional calculus tools to better describe and analyse systems and phenomena that display non-integer order dynamics or behaviours. Its applications span various fields where traditional calculus falls short in capturing the complexity and memory effects inherent in many natural and engineered systems. Fractional order partial differential equations enrich the traditional theory of PDEs by accommodating non-integer order derivatives, which are essential for describing systems with memory, long-range interactions, and complex dynamics across various scientific and engineering domains. Their applications continue to expand as researchers seek more accurate and comprehensive models for understanding and predicting real-world phenomena.

The classical Boussinesq equation was introduced by Joseph Boussinesq in 1871 to describe long waves in shallow water. In 1987, P. L. Sachdev and K. R. C. Nair were the first to propose fractional Boussinesq equation to account for fractional derivatives to capture better the memory effects and non-local interactions observed in many physical systems [6]–[9]. In particular, the time-space fractional Boussinesq equation is a notable example that arises in the context of nonlinear wave propagation and fluid dynamics. This equation combines fractional derivatives in both time and space variables, posing challenges for conventional analytical and numerical methods. The Boussinesq equation, originally proposed to describe long waves in shallow water, has been extended. The time-space fractional Boussinesq equation is expressed as follows:

$$D_t^{\alpha} u(x,t) = D_t^{\beta} u(x,t) + 3(u(x,t))^2{}_{xx} + (u(x,t))_{xxxx}, \tag{1}$$

$1 < \alpha, \beta < 2, t > 0, x \in (-L, L)$, D_t^{α}, and D_t^{β} denote the Caputo fractional derivatives concerning time (t) and space (x), respectively. This equation is significant in several fields due to its ability to model systems with non-local interactions and memory effects. These features are crucial for accurately describing a wide range of physical phenomena, including water waves [10], plasma physics [11], nonlinear optics [12], [13], Bose-Einstein condensates [14], and applications for groundwater flow [15].

In the literature, the existence and uniqueness of solutions of the Boussinesq equation have been studied [16]. Traveling wave solutions of the Boussinesq equation investigated by Alam *et al*. [17]. Stability analysis was conducted by Helal *et al*. [18]. Various researchers have studied several numerical methods to find a solution to the fractional Boussinesq equation. Particularly, finite element methods were proposed by Ramaswamy [19], spectral methods were used by Zhang *et al*. [20]. Takale [21] developed finite difference methods.

Recently, Fractional Boussinesq equations with non-local terms have been studied by Hu and Li [22]. Fractional Boussinesq equations with time-dependent coefficients are investigated by Y. Zhou and J. Wen-Wang Qi [23] developed the Adomian decomposition method, Botmart *et al*. [24] discussed the Natural decomposition approach, Yadav [25] used Crank-Nicolson finite difference scheme, Xu *et al*. [26] constructed fractional power series solution, Alyobi *et al*. [27] developed new approximate analytical solutions using the Laplace transform. The Atangana–Baleanu fractional derivative operator, Javeed *et al*. [28] adopt first integral method, Ali *et al*. [29] suggest modified Sardar-sub equation technique.

Due to the complexities introduced by fractional derivatives, traditional methods often struggle to provide exact solutions for the time-space fractional Boussinesq equation. This motivates the exploration of alternative analytical techniques, such as the homotopy perturbation method. This method, introduced by He [30], has proven effective in solving nonlinear differential equations by constructing a homotopy parameter and introducing perturbation terms that enable the generation of accurate analytical approximations [30]–[34].

The structure of the paper is as follows: Section 2 provides a brief overview of the fractional calculus concepts and the time-space fractional Boussinesq equation formulation. Section 3 outlines the homotopy perturbation method and its application to the time-space fractional Boussinesq equation. Section 4 presents the analytical results obtained using the homotopy perturbation method and discusses their implications. Finally, Section 5 concludes with a summary of findings and avenues for future research.

2. Fractional Order Derivatives and Integrals

Definition 1: The Riemann-Liouville fractional integral of a function $u(x)$ of order $\alpha > 0$ is denoted by $I^{\alpha}u(x)$ and is defined as follows [35]:

$$I^{\alpha}u(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}u(t)\,dt \tag{2}$$

Definition 2: The Caputo derivative of a function $u(t)$ of order $\alpha > 0$ is denoted by $\frac{\partial^{\alpha}u(t)}{\partial t^{\alpha}}$ and is defined as follows [35]:

$$\frac{\partial^{\alpha}u(t)}{\partial t^{\alpha}} = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\tau)^{n-\alpha-1}\frac{d^n}{d\tau^n}u(\tau)\,d\tau \tag{3}$$

where Γ is the gamma function, n is the smallest integer greater than α, a is the lower limit of integration, and $\frac{d^n}{d\tau^n}$ denotes the n-th derivative of u with respect to τ.

Remark:

i) $I^{\alpha}(t^s) = \frac{\Gamma(s+1)}{\Gamma(\alpha+s+1)}t^{\alpha+s}$ for $s > -1, \alpha \geq 0$

ii) $I^{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}(u(x)) = u(x) - \sum_{k=0}^{r-1}u^{(k)}\left(0^{+}\right)\frac{x^k}{k!}, x > 0, \alpha \geq 0$

iii) $I^{\alpha}I^{\beta}(u(t)) = I^{\alpha+\beta}(u(t))$ for $\alpha, \beta \geq 0$

3. Fractional Homotopy Perturbation Method

The method constructs a homotopy $v(x,p) : X \times [0,1] \rightarrow Y$ that continuously transforms the original problem into a simpler problem. Here, $p \in [0,1]$ is an embedding parameter, X is the original space, and Y is the target space.

Consider the generalized nonlinear differentiation equations as follows:

$$A(u) - f(x) = 0 \tag{4}$$

where A is a nonlinear operator, u is the unknown function, and $f(x)$ is a known function, we construct a homotopy $H(v,p)$ in the form:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(x)] = 0 \tag{5}$$

where L is a linear operator extracted from the operator A, u_0 is an initial approximation of u, and p is the embedding parameter. When $p = 0$, $H(v, 0) = L(v) - L(u_0) = 0$, which is a simple problem to solve. When $p = 1$, $H(v, 1) = A(v) - f(x) = 0$, which is the original problem. The solution v is expressed as a series in p:

$$v = v_0 + pv_1 + p^2 v_2 + \cdots \tag{6}$$

As p approaches 1, the series converges to the solution of the original problem:

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots \tag{7}$$

4. Numerical Solution to the Time-Space Fractional Boussinesq Equation

Consider the time-space fractional Boussinesq equation:

$$D_t^{\alpha} u(x, t) = D_t^{\beta} u(x, t) + 3(u(x, t))^2{}_{xx} + (u(x, t))_{xxxx}\,, \tag{8}$$

with:

$$u_0 = \frac{1}{2}\left(c^2 - 1\right) x^2 \ \sec h^2 \left(\frac{1}{2}\sqrt{c^2 - 1}\right) \tag{9}$$

To apply the homotopy perturbation technique for the time-space fractional Boussinesq equation, we construct a homotopy $H(v, p)$ As follows:

$$\begin{aligned} H(v, p) = (1 - p)\left[D_t^{\alpha} v(x, t) - D_t^{\alpha} u(x, 0)\right] \\ + p\left[D_t^{\alpha} v(x, t) - D_t^{\beta} v(x, t) - 3(v(x, t))^2{}_{xx} - (v(x, t))_{xxxx}\right] = 0 \end{aligned} \tag{10}$$

where $u_0 = u(x, 0)$. Now, express the solution v for the time-space fractional Boussinesq equation as a series in p as:

$$v(x, t) = v_0 + pv_1 + p^2 v_2 + \cdots \tag{11}$$

Substitute this series into the homotopy $H(v, p)$, and by equating the coefficients of like powers of p to zero, we obtain a series of linear equations that can be solved sequentially as follows:

$$\begin{aligned} p^0 &: v_0 = u(x, 0) \\ p^1 &: v_1 = I^{\alpha}\left(D_x^{\beta} v_0 + 3(v_0)^2{}_{xx} + (v_0)_{xxxx}\right) \\ p^2 &: v_2 = I^{\alpha}\left(D_x^{\beta} v_1 + 6\,(v_0 v_1)_{xx} + (v_1)_{xxxx}\right) \\ p^3 &: v_3 = I^{\alpha}\left(D_x^{\beta} v_2 + 3(v_1)^2{}_{xx} + 6\,(v_0 v_2)_{xx} + (v_2)_{xxxx}\right) \\ p^4 &: v_4 = I^{\alpha}\left(D_x^{\beta} v_3 + 6\,(v_1 v_2)_{xx} + 6\,(v_0 v_3)_{xx} + (v_3)_{xxxx}\right) \\ &\vdots \end{aligned} \tag{12}$$

and so on. Solving the above system of equations with $u_0 = \frac{1}{2}\left(c^2 - 1\right) x^2\, sech^2\left(\frac{1}{2}\sqrt{c^2 - 1}\right)$, we obtain the values of $v_0, v_1, \cdots$ for different values for α and β.

For $\alpha = 2, \beta = 2$, we obtain an approximate solution for the time-space fractional Boussinesq equation as follows:

$$u(x, t) = \frac{1}{2}\left(c^2 - 1\right) x^2\, sech^2\left(\frac{1}{2}\sqrt{c^2 - 1}\right) + \frac{9\left(c^4 - 2c^2 + 1\right) t^2 x^2 + \left(c^2 - 1\right) t^2 \, cosh^2\left(\frac{1}{2}\sqrt{c^2 - 1}\right)}{2\, cosh^4\left(\frac{1}{2}\sqrt{c^2 - 1}\right)}$$

$$+\frac{27\left(c^6-3c^4+3c^2-1\right)t^4x^2+2\left(c^4-2c^2+1\right)t^4\cosh^2\left(\frac{1}{2}\sqrt{c^2-1}\right)}{2\cosh^6\left(\frac{1}{2}\sqrt{c^2-1}\right)}$$
$$+\frac{81\left(c^8-4c^6+6c^4-4c^2+1\right)t^6x^2+4\left(c^6-3c^4+3c^2-1\right)t^6\cosh^2\left(\frac{1}{2}\sqrt{c^2-1}\right)}{2\cosh^8\left(\frac{1}{2}\sqrt{c^2-1}\right)}+\cdots \quad (13)$$

Also, we obtain solutions for the time-space fractional Boussinesq equation at $\alpha = 1.9,\ \beta = 1.9$,

$$u(x,t)=\frac{1}{2}\left(c^2-1\right)x^2\,sech^2\left(\frac{1}{2}\sqrt{c^2-1}\right)$$
$$+\frac{\left(0.59c^2-0.59\right)t^{1.9}x^2\cosh^2\left(\frac{1}{2}\sqrt{c^2-1}\right)+\left(4.92c^4-9.85c^2+4.92\right)t^{1.9}x^2}{\cosh^4\left(\frac{1}{2}\sqrt{c^2-1}\right)}$$
$$+\cdots \quad (14)$$

The exact solution for the given problem at $\alpha = 2,\ \beta = 2$ is:

$$u(x,t)=\frac{1}{2}\left(c^2-1\right)\left(ct-x^2\right)x^2\,sech^2\left(\frac{1}{2}\sqrt{c^2-1}\right) \quad (15)$$

In Table I, we contrast the exact solution with the approximate solution for the time-space fractional Boussinesq equation, revealing a remarkable closeness between the two.

We compare the estimated approximated solution for the time-space fractional Boussinesq equation with an exact solution at $0 \le x \le 10,\ \alpha = 2,\ \beta = 2,\ c = 2,\ t = 0.02$ in Fig. 1 and observe that the approximate solution is closed to the exact solution.

In Fig. 2, we examined the behaviour of solutions for the time-space fractional Boussinesq equation at $0 \le x \le 10,\ c = 2,\ t = 0.03$ for different values of $\alpha = \beta = 2,\ 1.9,\ 1.8$ and observed that the obtained solution is traveling wave solution converges towards the solution for $\alpha = 1$.

Simulations of the solution for the time-space fractional Boussinesq equation at $t = 0.05,\ c = 2,\ \alpha = 1.9,\ \beta = 1.8$ is displayed in Fig. 3.

TABLE I: Absolute Error between Exact Solution and Approximate Solution, $u(x,t)$ for $\alpha = 2$, $\beta = 2$, and $c = 2$

$x \rightarrow$	0.01	0.02	0.03	0.04	0.05
$t = 0.01$	5.7822×10^{-5}	3.7349×10^{-4}	6.8654×10^{-4}	9.9862×10^{-4}	1.3102×10^{-3}
$t = 0.02$	2.4984×10^{-4}	8.8723×10^{-4}	1.5209×10^{-4}	2.1531×10^{-3}	1.3730×10^{-3}
$t = 0.03$	1.3730×10^{-3}	3.7982×10^{-4}	5.9230×10^{-4}	1.5565×10^{-3}	2.5175×10^{-3}
$t = 0.04$	2.8731×10^{-3}	1.5199×10^{-3}	2.0306×10^{-4}	1.1002×10^{-3}	2.3977×10^{-3}
$t = 0.05$	4.9002×10^{-3}	3.1733×10^{-3}	1.5021×10^{-3}	1.4845×10^{-4}	1.7901×10^{-3}

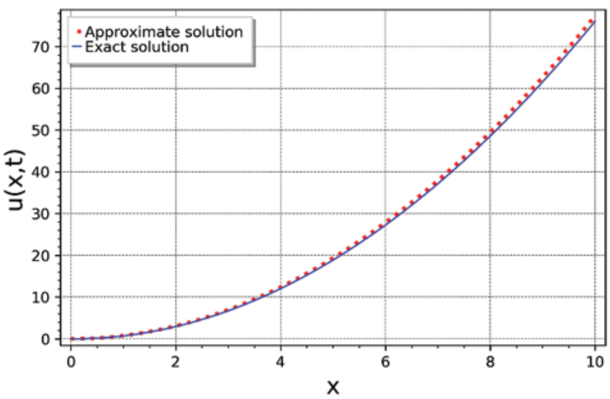

Fig. 1. Comparison of approximate solution with exact solution at $0 \le x \le 10$, $t = 0.05$, $\alpha = 2$, and $\beta = 2$, $c = 2$, and $t = 0.02$.

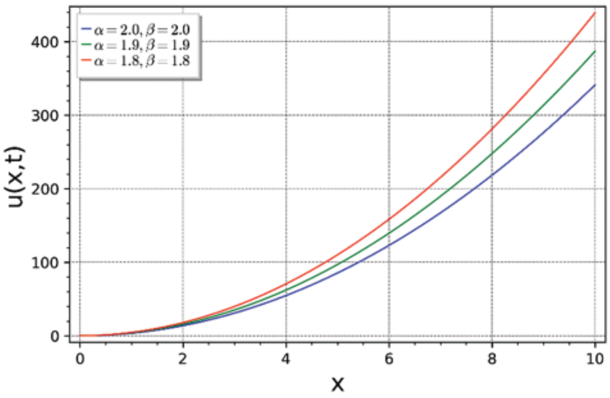

Fig. 2. Behaviour of solutions for $0 \leq x \leq 10$, $c = 2$, $t = 0.03$.

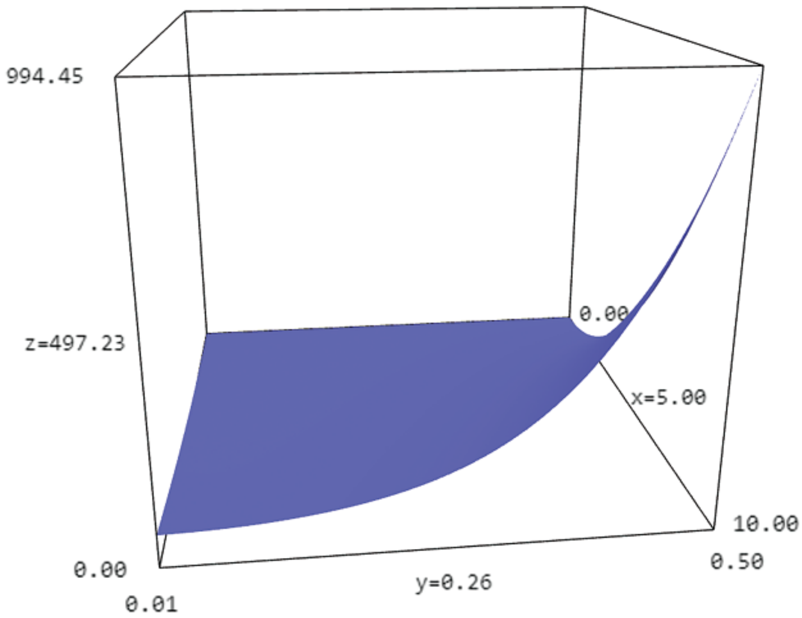

Fig. 3. Simulations of the solution at $t = 0.05$, $c = 2$, $\alpha = 1.9$, and $\beta = 1.8$.

5. Conclusion

Using the fractional homotopy perturbation method, we solved the time-space fractional Boussinesq equation. Analytical solutions obtained through the homotopy perturbation technique were compared and validated against the exact solution. Results demonstrated that this method provides efficient and accurate solutions for the time-space fractional Boussinesq equation. The analytical solutions offer insights into how fractional derivatives influence wave dynamics, emphasizing the importance of considering memory and hereditary effects in modeling real-world phenomena. We observe the traveling wave solutions for the time-space fractional Boussinesq equation, representing the solitary wave solution type.

Conflict of Interest

The authors declare that they do not have any conflict of interest.

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