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RESEARCH ARTICLE

EllipsoHyperbola A Common Approach that Joins the Conic Sections in 2D and 3D Space

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ABSTRACT

This work aspires to interactively reveal through the use of the GeoGebra Software the relationship between the Conic Sections in 3D and the 2D symmetric forms of Conic Sections around O in a coordinate system Oxy, that is Circle, Ellipse, axis x′x, and Hyperbola, showing that these Conic Sections arise from the same algebraic formula and therefore have common characteristics. This manuscript includes short research on the four kinds of Conic Sections through a common approach that joins them in the two-but also in three-dimensional space, revealing the role of the slope of the Generator line of the conic surface and the role of the slope of the cutting plane in their equations.

Keywords: Conic sections, Ellipse, Ellipse-Hyperbola, Hyperbola.

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1. Introduction

This work of mine is a continuation and extension of the work on "Ellipse-Hyperbola" [1] which was presented at the 37th Pan-Hellenic Mathematical Education Conference of the Hellenic Mathematical Society in November 2023, in Argos-Naples. It has been taught to my respective students at least since 2010, as it can easily be certified from a relevant GeoGebra [2] workshop file that has been uploaded since then in my Blog [3].

In this Laboratory presentation, the following topics are investigated algebraically and graphically, i.e., with the interaction of algebraic and computer methods:

1. The hidden relationship between Ellipse and Hyperbola, which is none other than their common equation, as I call it the "Ellipse-Hyperbolic" equation, as an explanation of their similarities.
2. The explanation of the above is from the position of the "Ellipse-Hyperbola" as a Generalized Conic Section, which results from the section of a Conic Surface with a different inclined plane, in the 3 Dimension space.

For this laboratory presentation, a basic familiarity with the GeoGebra software environment is required, as well as knowledge of the corresponding chapters of Mathematics and some knowledge of 3D Analytical Geometry. Therefore, no extensive bibliography is required beyond textbooks and some authoritative online books, given in the bibliography. It is also not necessary to install any software, since everything has been uploaded to the repository [2] offered by the GeoGebra software and anyone can, through the interfaces (links) of the presentation, execute and verify the mentioned.

It is proposed to be presented by the Mathematician in the classroom or a computer laboratory in a Mathematic Orientation course of the 2nd class of Greek High School, during the corresponding Conic Sections course, where the students can try and test the mentioned. Then a comparative table of Conic Sections can be constructed by the students and a worksheet can be filled up instead of writing an exam test.

The whole process to serve its educational role is recommended to be done in a good atmosphere in the form of a game, stimulating the students' interest and motivating them for experimentation, research and action.

For the presentation it will be needed:

1. A Computer
2. A Projector connected to the computer
3. Internet connection.

2. Teaching Approach

2.1. *The Experiment (https://www.GeoGebra.org/m/xcqq3fdj)*

The purpose of this object is for the student to discover in an interactive GeoGebra environment that mainly these two conic sections (we consider the circle to be a sub case of the Ellipse when its points coincide) are described by a single equation, that is why they have common characteristics, such as similar definitions and equations, the existence of two points, the eccentricity with $\varepsilon = \gamma/\alpha$, the base rectangle, their reflective property, etc., which are differentiated, when the values of the parameters α and γ of their equation exceed one another's value, i.e., Ellipse when $\alpha > \gamma$ (Fig. 1) and Hyperbola when $\alpha < \gamma$ (Fig. 2).

Indeed, for any values of $\alpha, \gamma \in (0, +\infty)$, with $\alpha < \gamma$ or $\alpha > \gamma$. Because for $\gamma = 0$ we have a circle, the horizontal Ellipse $C\colon \dfrac{x^2}{\alpha^2} + \dfrac{y^2}{\beta^2} = 1$ for $\beta^2 = \alpha^2 - \gamma^2$ and $\alpha > \gamma$ and Focus points E(γ, 0) and E′(−γ,

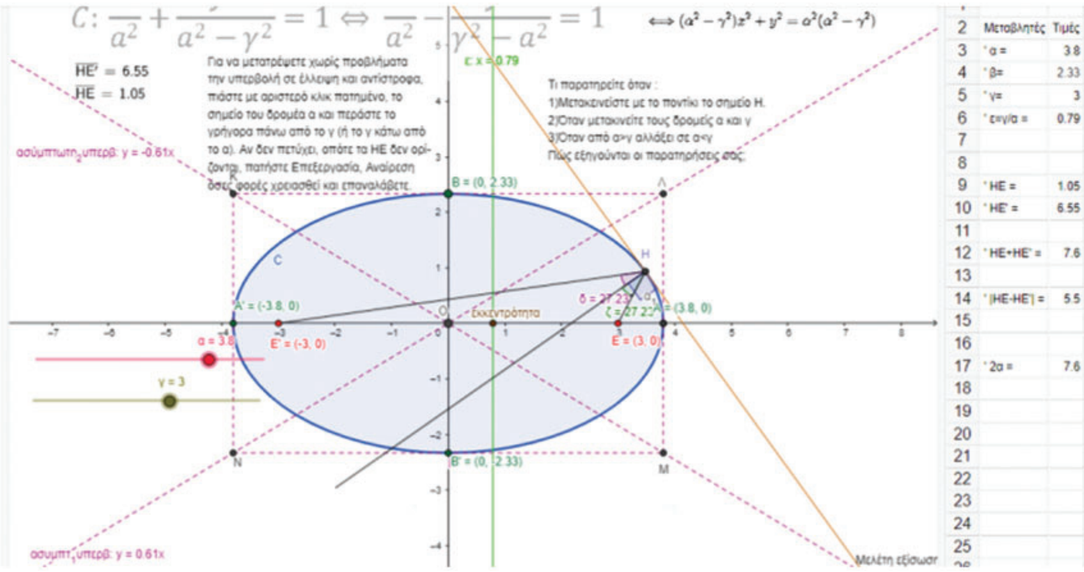


Fig. 1. Ellipse on the previous GeoGebra link when $\alpha > \gamma > 0$.

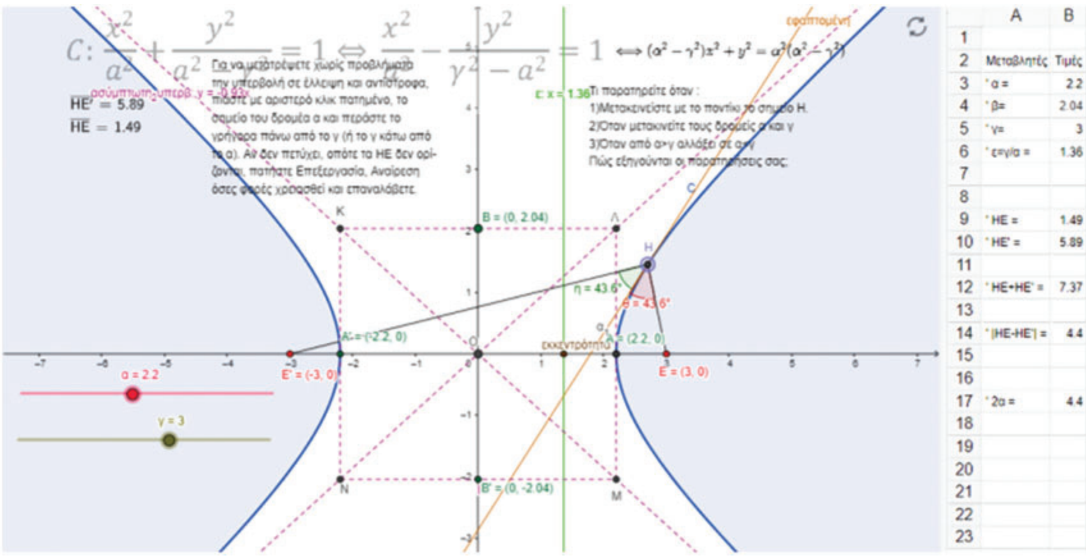


Fig. 2. Hyperbola on the previous GeoGebra link when $\gamma > \alpha > 0$.

0) it can be written as:

$$C: \frac{x^2}{a^2} + \frac{y^2}{a^2 - \gamma^2} = 1 \Longleftrightarrow \frac{x^2}{a^2} - \frac{y^2}{\gamma^2 - \alpha^2} = 1 \Longleftrightarrow \frac{x^2}{a^2} - \frac{y^2}{\beta'^2} = 1,$$

for $\beta'^2 = \gamma^2 - \alpha^2$ and $\alpha < \gamma$, so Horizontal Hyperbola, having also Focus points E(γ, 0) and E′($-\gamma$, 0).

Similarly, the vertical Ellipse with Focus points E(0, γ) and E′(0, $-\gamma$) $C: \frac{y^2}{a^2} + \frac{x^2}{\beta^2} = 1$, having $\beta^2 = \alpha^2 - \gamma^2$, and $\alpha > \gamma$, is

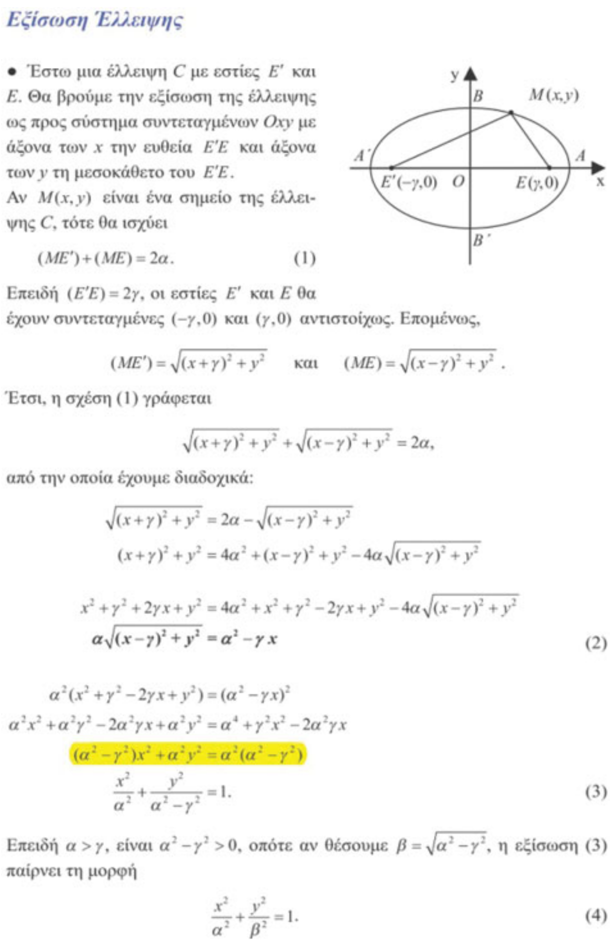
$$C: \frac{y^2}{\alpha^2} + \frac{x^2}{\alpha^2 - \gamma^2} = 1 \Longleftrightarrow \frac{y^2}{\alpha^2} - \frac{x^2}{\gamma^2 - \alpha^2} = 1 \Longleftrightarrow \frac{y^2}{\alpha^2} - \frac{x^2}{\beta'^2} = 1,$$

for $\beta'^2 = \gamma^2 - \alpha^2$ and $\alpha < \gamma$, that is vertical Hyperbola having also Focus points E(0, γ) and E′(0, $-\gamma$).

Regarding the eccentricities, in all cases $\varepsilon = \gamma/\alpha$, and in both Ellipses $\alpha > \gamma$, so $0 < \varepsilon < 1$, while in their respective Hyperbolas $\alpha < \gamma$ therefore $\varepsilon > 1$.

Therefore, in both cases by taking the equation of Ellipse and changing the value of α so that from $\alpha > \gamma$ it becomes $\alpha < \gamma$, the eccentricity changes from $0 < \varepsilon < 1$ to $\varepsilon > 1$ and the Ellipse is converting into Hyperbola with corresponding points. This fact is verified very easily and experimentally with graphic software such as GeoGebra.

Finally, through this spreadsheet of GeoGebra, all the properties of the Ellipse and Hyperbole can be studied also by the help of the right accounting sheet, such as the stability of the sum in the Ellipse and of the absolute difference in the Hyperbola of the distances of their points from their Focus points, the eccentricity $0 < \varepsilon < 1$ for the Ellipse and $\varepsilon > 1$ for the Hyperbola, their reflective properties, their basic rectangular, the asymptotes, etc.



Εξίσωση Έλλειψης

- Έστω μια έλλειψη C με εστίες E' και E. Θα βρούμε την εξίσωση της έλλειψης ως προς σύστημα συντεταγμένων Oxy με άξονα των x την ευθεία $E'E$ και άξονα των y τη μεσοκάθετο του $E'E$.

Αν $M(x,y)$ είναι ένα σημείο της έλλειψης C, τότε θα ισχύει

$$(ME') + (ME) = 2\alpha. \quad (1)$$

Επειδή $(E'E) = 2\gamma$, οι εστίες E' και E θα έχουν συντεταγμένες $(-\gamma, 0)$ και $(\gamma, 0)$ αντιστοίχως. Επομένως,

$$(ME') = \sqrt{(x+\gamma)^2 + y^2} \quad \text{και} \quad (ME) = \sqrt{(x-\gamma)^2 + y^2}.$$

Έτσι, η σχέση (1) γράφεται

$$\sqrt{(x+\gamma)^2 + y^2} + \sqrt{(x-\gamma)^2 + y^2} = 2\alpha,$$

από την οποία έχουμε διαδοχικά:

$$\sqrt{(x+\gamma)^2 + y^2} = 2\alpha - \sqrt{(x-\gamma)^2 + y^2}$$

$$(x+\gamma)^2 + y^2 = 4\alpha^2 + (x-\gamma)^2 + y^2 - 4\alpha\sqrt{(x-\gamma)^2 + y^2}$$

$$x^2 + \gamma^2 + 2\gamma x + y^2 = 4\alpha^2 + x^2 + \gamma^2 - 2\gamma x + y^2 - 4\alpha\sqrt{(x-\gamma)^2 + y^2}$$

$$\boldsymbol{\alpha\sqrt{(x-\gamma)^2 + y^2} = \alpha^2 - \gamma x} \quad (2)$$

$$\alpha^2(x^2 + \gamma^2 - 2\gamma x + y^2) = (\alpha^2 - \gamma x)^2$$

$$\alpha^2 x^2 + \alpha^2\gamma^2 - 2\alpha^2\gamma x + \alpha^2 y^2 = \alpha^4 + \gamma^2 x^2 - 2\alpha^2\gamma x$$

$$(\alpha^2 - \gamma^2)x^2 + \alpha^2 y^2 = \alpha^2(\alpha^2 - \gamma^2)$$

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\alpha^2 - \gamma^2} = 1. \quad (3)$$

Επειδή $\alpha > \gamma$, είναι $\alpha^2 - \gamma^2 > 0$, οπότε αν θέσουμε $\beta = \sqrt{\alpha^2 - \gamma^2}$, η εξίσωση (3) παίρνει τη μορφή

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1. \quad (4)$$

Fig. 3. Page 102 of the Greek school textbook [4].

2.2. The Explanation

Let C be an Ellipse and C′ a Hyperbola with their focus points be E′ and E. The equation of both Ellipse and Hyperbola in terms of Oxy coordinates system with axis of x the line E′ E and axis of y the mid-vertical of E′ E with E′(−γ, 0) and E(γ, 0).

As we can see in Fig. 3, any Ellipse C must consist of points M(x, y) of which the sum of their distances from E′ and E is constantly 2α, so must be (ME) + (ME′) = 2α.

So, $\sqrt{(x+\gamma)^2+y^2}+\sqrt{(x-\gamma)^2+y^2}=2\alpha \Leftrightarrow \left(\sqrt{(x+\gamma)^2+y^2}\right)^2=\left(2\alpha-\sqrt{(x-\gamma)^2+y^2}\right)^2 \Leftrightarrow$

$$\alpha\sqrt{(x-\gamma)^2+y^2}=\alpha^2-\gamma x \Leftrightarrow \alpha^2\left(x^2+\gamma^2-2\gamma x+y^2\right)=(\alpha^2-\gamma x)^2,$$

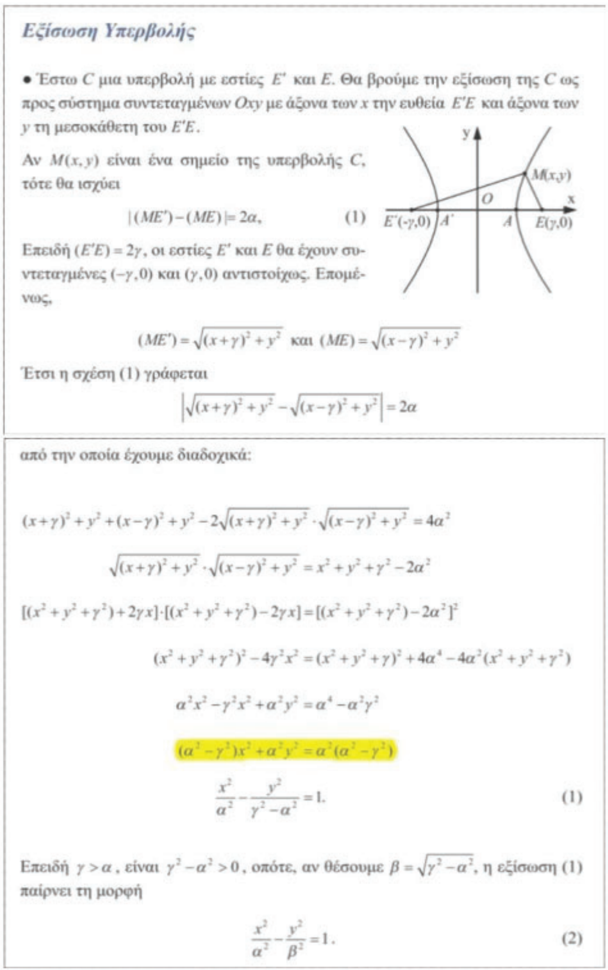
that finally gives [4]:

$$\left(\alpha^2-\gamma^2\right)x^2+\alpha^2y^2=\alpha^2\left(\alpha^2-\gamma^2\right) \tag{1}$$

On the other hand, as we can see in Fig. 4, any Hyperbola C′ must consist of points that the absolute difference of their distances from E′ and E is constant 2α. Therefore, for any point M(x, y) of C′, it will be $\left|(ME)-(ME')\right|$ = 2α, so

$$\left|\sqrt{(x+\gamma)^2+y^2}-\sqrt{(x-\gamma)^2+y^2}\right|=2\alpha \Leftrightarrow \left(\sqrt{(x+\gamma)^2+y^2}-\sqrt{(x-\gamma)^2+y^2}\right)^2=4\alpha^2 \Leftrightarrow$$
$$\sqrt{(x+\gamma)^2+y^2}\sqrt{(x-\gamma)^2+y^2}=x^2+y^2+\gamma^2-2\alpha^2 \Leftrightarrow \alpha^2x^2-\gamma^2x^2+\alpha^2y^2=\alpha^4-\alpha^2\gamma^2,$$

that also gives [4] the same from above (1) that I have deliberately also marked up above in yellow on the copies of the pages of the Greek school textbook, which I present in Figs. 3 and 4, for being noticeable, as nothing is written or told anywhere yet about those obviously identical equations.



Εξίσωση Υπερβολής

• Έστω C μια υπερβολή με εστίες E' και E. Θα βρούμε την εξίσωση της C ως προς σύστημα συντεταγμένων Oxy με άξονα των x την ευθεία $E'E$ και άξονα των y τη μεσοκάθετη του $E'E$.

Αν $M(x,y)$ είναι ένα σημείο της υπερβολής C, τότε θα ισχύει

$$|(ME')-(ME)|=2\alpha, \tag{1}$$

Επειδή $(E'E)=2\gamma$, οι εστίες E' και E θα έχουν συντεταγμένες $(-\gamma,0)$ και $(\gamma,0)$ αντιστοίχως. Επομένως,

$$(ME')=\sqrt{(x+\gamma)^2+y^2} \text{ και } (ME)=\sqrt{(x-\gamma)^2+y^2}$$

Έτσι η σχέση (1) γράφεται

$$\left|\sqrt{(x+\gamma)^2+y^2}-\sqrt{(x-\gamma)^2+y^2}\right|=2\alpha$$

από την οποία έχουμε διαδοχικά:

$$(x+\gamma)^2+y^2+(x-\gamma)^2+y^2-2\sqrt{(x+\gamma)^2+y^2}\cdot\sqrt{(x-\gamma)^2+y^2}=4\alpha^2$$

$$\sqrt{(x+\gamma)^2+y^2}\cdot\sqrt{(x-\gamma)^2+y^2}=x^2+y^2+\gamma^2-2\alpha^2$$

$$[(x^2+y^2+\gamma^2)+2\gamma x]\cdot[(x^2+y^2+\gamma^2)-2\gamma x]=[(x^2+y^2+\gamma^2)-2\alpha^2]^2$$

$$(x^2+y^2+\gamma^2)^2-4\gamma^2x^2=(x^2+y^2+\gamma)^2+4\alpha^4-4\alpha^2(x^2+y^2+\gamma^2)$$

$$\alpha^2x^2-\gamma^2x^2+\alpha^2y^2=\alpha^4-\alpha^2\gamma^2$$

$$(\alpha^2-\gamma^2)x^2+\alpha^2y^2=\alpha^2(\alpha^2-\gamma^2)$$

$$\frac{x^2}{\alpha^2}-\frac{y^2}{\gamma^2-\alpha^2}=1. \tag{1}$$

Επειδή $\gamma>\alpha$, είναι $\gamma^2-\alpha^2>0$, οπότε, αν θέσουμε $\beta=\sqrt{\gamma^2-\alpha^2}$, η εξίσωση (1) παίρνει τη μορφή

$$\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}=1. \tag{2}$$

Fig. 4. Pages 114–115 of the Greek school textbook [4].

By turning it to English in order to connect it with the rest script it becomes to:

$$\left(a^2 - c^2\right) x^2 + a^2 y^2 = a^2 \left(a^2 - c^2\right) \tag{2}$$

So, this is actually the common equation for both Ellipse and Hyperbola, as for a > c gives $\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$, and for a < c gives $\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$. But not only the two of them, as for c = 0 (1) $\Longleftrightarrow x^2 + y^2 = a^2$, equation of a Circle of radius a.

Also, for a = c (1) $\Longleftrightarrow y = 0$, which indicates the x′x axis. Besides as we can see on the Figs. 5–7 and test in; https://www.geogebra.org/m/zzpwsya6, when we change the value of parameter p in Parabola [4] C: $y^2 = 2px$ from p > 0 to p < 0 the graph tends to cover the semi-axis Ox and then for p < 0 it turns from right (Fig. 5) to the left (Fig. 6) side of axis y′y. Though in the middle for p = 0 (Fig. 7) it covers the whole of axis x′x: y = 0. So, x′x: y = 0 appears to be a kind of a symmetric around O(0,0) disintegrated parabola.

In the following section, we will show that in this case x′x axis is the projection on the plane Oxy of an intermediate degenerated parabola, where the plane parallel to the origin line of the 3D double

Fig. 5. Parabola $y^2 = 2$ px when p > 0.

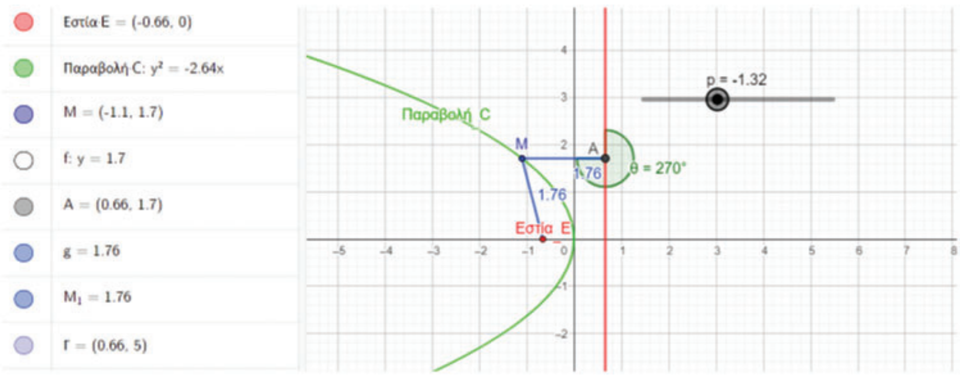


Fig. 6. Parabola $y^2 = 2$ px when p < 0.

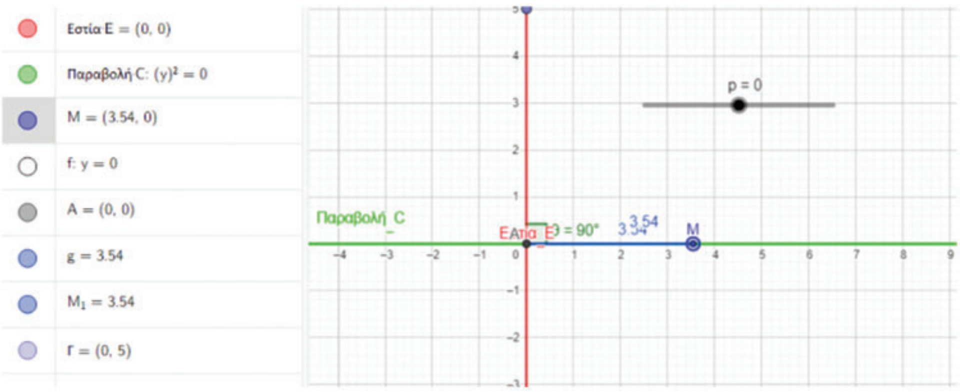


Fig. 7. DegeneratedParabola $y^2 = 2$ px when p = 0 (axis x′x).

conic surface (whose intersection with these cones gives the parabola) passes then through the vertex O and touches on a straight line the two conical surfaces. As (1) produces only Conic Sections having a symmetry centre at the point O(0,0) it produces axis x′x which is the only parabola with O(0,0) as the symmetry centre.

3. The Relationship of the "Ellipse-Hyperbola" as a Generalized Conic Section in the Space of 3 Dimensions

3.1. *Experimentally (https://www.geogebra.org/m/qfpfzjjk)*

As it is known from Stereometry, in an orthonormal system of three axes Oxzy in the space of three dimensions, the double conical surface is formed by the rotation of a line (Fig. 3), the Generator line (here purple dashed line) that passes through O(0,0,0) and generates the double conic surface by its imaginary circular rotation at a constant angle φ around the vertical (blue) axis z′z. The complementary φ angle $\theta = 90° - \varphi$ has an inclination $\alpha = \tan\theta$, where θ is the angle of the Generator line and the gray level Oxy, which changes values from the corresponding bar (slider α), opening or closing the cone on the GeoGebra sheet of previous link and the next Fig. 8.

The gray plane Oxy is the plane of the x′x (red) and y′y (green) axes, while the yellow plane P intersects the double conic surface CS by a conic section of the same or different type depending on the slope of $\gamma = \tan\theta'$, where θ is the angle of the yellow plane P and the gray plane Oxy, which changes with the corresponding bar (slider γ), and the bar δ (slider δ) where δ is the point of intersection of the yellow plane P with the vertical (blue) z axis ′z.

Next to the 3D figure we see on the right the 2D projection of the (purple) conic section being formed on the horizontal (gray) Oxy plane, where the yellow plane P is parallel to the y′y axis. This does not affect the investigation, because we can define the horizontal axis system Oxy so that the y′y axis to be the parallel line to the yellow plane P belonging to the perpendicular to the z′z plane and passing through the point O.

Notice in Fig. 8, that when $0 < \gamma < \alpha$ we have an Ellipses, becoming to Circle if $\gamma = 0$, as in Fig. 9, the gradient $\gamma = 0 = \tan 0°$ so that the yellow plane becomes horizontal.

For $\gamma > \alpha > 0$, as you can see in Fig. 10, we have Hyperbolas.

Also, in the transitional case where momentarily $\gamma = \alpha$ we have Parabola, as we can see in Fig. 11. Notice that the colour of the right 2D right graph is then changed from purple to light blue in case of Parabola as the equation of the reddish graph is:

$$\mathrm{C}:\ \frac{\left(x-\frac{\delta}{\beta^2}\gamma\right)^2}{\frac{\delta^2}{\beta^4}\alpha^2}+\frac{y^2}{\frac{\delta^2}{\beta^4}\beta^2}=1 \tag{3}$$

$$\text{or the equivalent } \mathrm{C}:\ \frac{\left(x+\frac{\delta}{\beta^2}\gamma\right)^2}{\frac{\delta^2}{\beta^4}\alpha^2}-\frac{y^2}{\frac{\delta^2}{\beta^4}\beta^2}=1 \tag{4}$$

$$\text{and for the light blue one is: } (\alpha^2-\gamma^2)x^2+\alpha^2y^2-2\gamma\delta x=\delta^2 \tag{5}$$

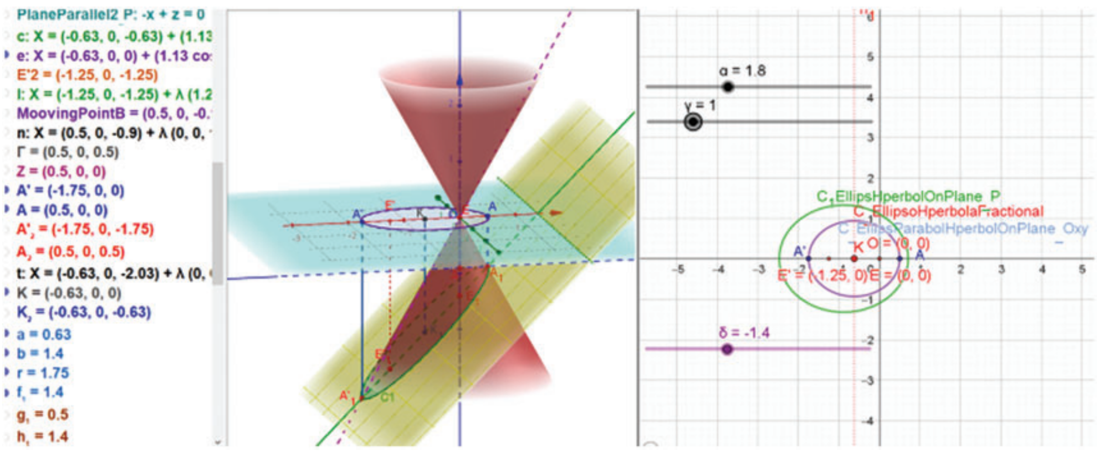


Fig. 8. Plane P//y′y cutting ellipse on the conic surface CS.

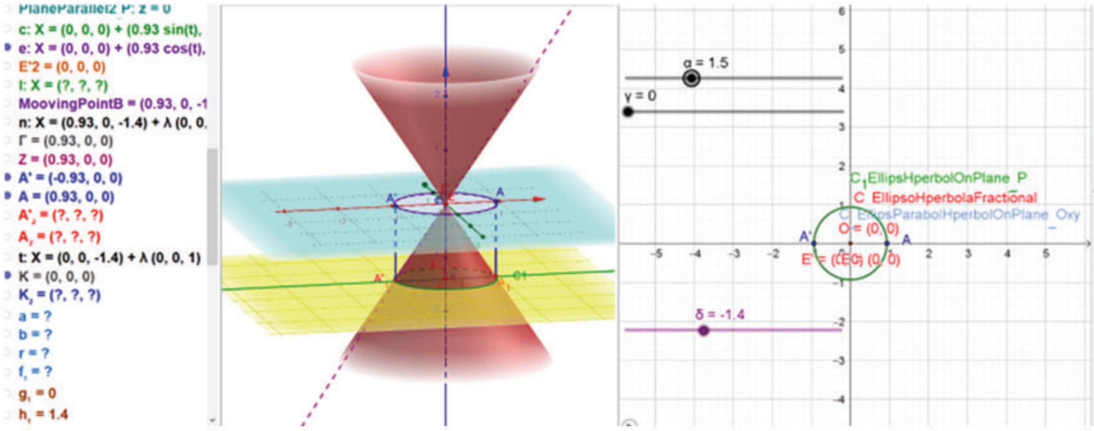


Fig. 9. Plane P//Oxy cutting circle on the conic surface CS.

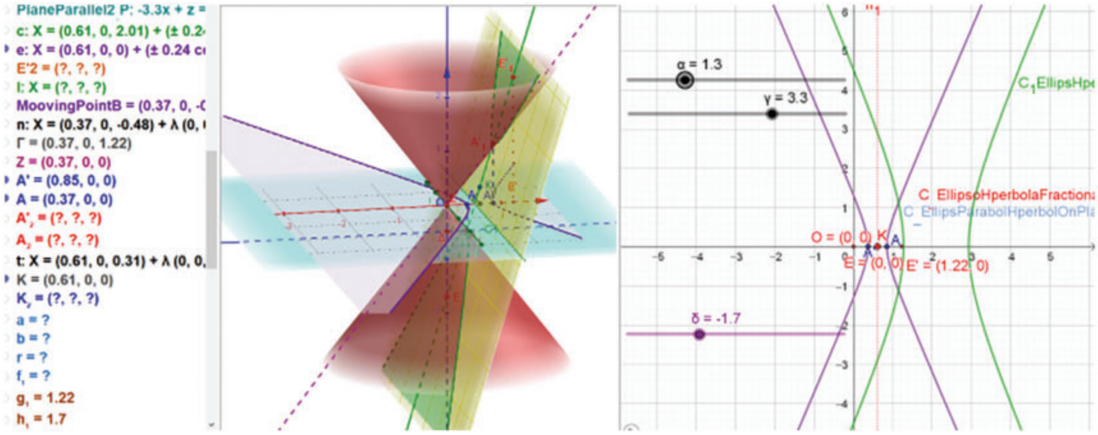


Fig. 10. Plane P//y′y cutting hyperbola on the conic surface CS.

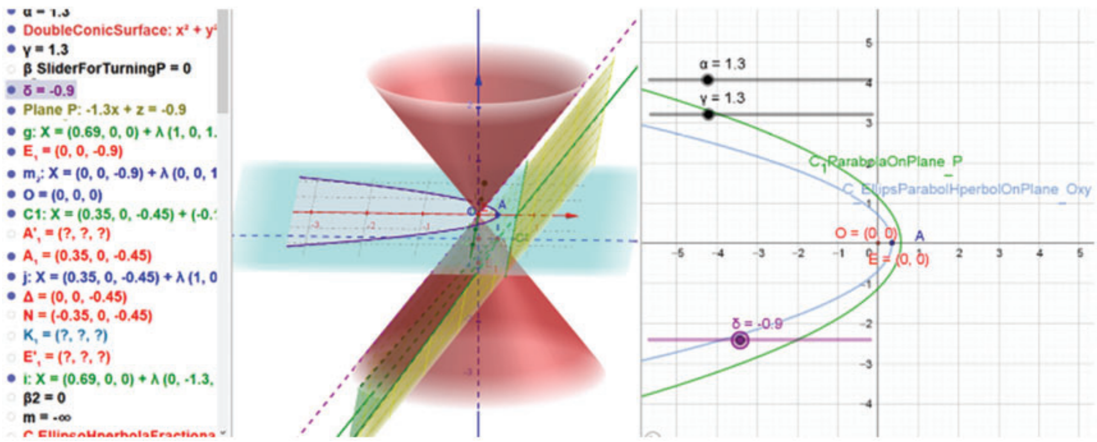


Fig. 11. Plane P parallel to a generator line cutting parabola on the conic surface CS.

(all (3)–(5) are justified in the following text).

Notice also that then the yellow plane is parallel to the Generator line of the conic surface and that it intersects only one cone, either the lower or the upper one, except when $\delta = 0$, when as we observe in the next Fig. 12, that the yellow plane touches both conic surfaces along the Generator straight line of the double conic surface CS.

Fig. 12 seems to be the case of the previous paragraph (II. B. iii), where we momentarily have $\alpha = \gamma$ and between Ellipses and Parabolas appears the x′x axis, which as we said there, is the case of the degenerated symmetric around O(0,0) Parabola.

Here we should emphasize that the names I gave to the variables α and γ for the bars, they have been chosen to remind the variables a and cin the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2 - c^2} = 1$ of Ellipses and the equivalent

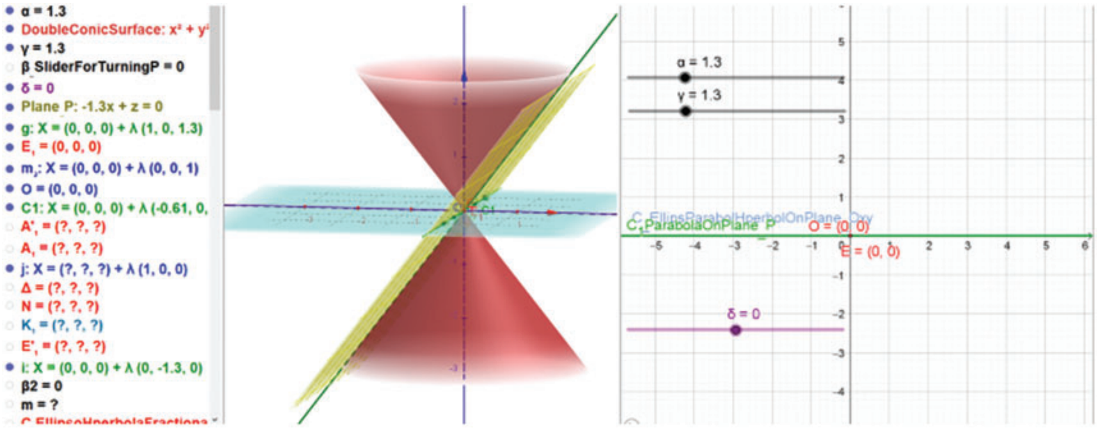


Fig. 12. Plane P touching conic surface CS on a generator line.

equation $\dfrac{x^2}{a^2}-\dfrac{y^2}{c^2-a^2}=1$ of Hyperbolas, because they similarly transform the conic sections from one to the other, although they are actually different from them. (Their connection will be later justified in this manuscript).

We also notice that always one of their Focus points of any kind of the materialised conic sections is the point where the vertical axis intersects the yellow plane, so it is the point $E_1(0,0,\delta)$.

In the case of the Circle where the two points coincide at its centre the vertical axis passes through the centre of the Circle, and in the case of Parabola the Focus point is again the intersection of the vertical axis with the yellow plane, and the other point is disappeared at infinity.

3.2. *Theoretically*

For simplicity reason we will reduce our study in case that the cutting (yellow) plane intersects the lower conic surface and the z′z axis under 0 at $z = \delta < 0$.

Having Generator line's inclination $\alpha \geq 0$, the equation of the double conic surface is given [5] by the formula:

$$CS\colon \alpha^2x^2+\alpha^2y^2=z^2 \tag{6}$$

while the equation of parallel to the y′y axis of yellow plane is

$$\mathrm{P}\colon \gamma\,\mathrm{x}+\delta=\mathrm{z} \tag{7}$$

(Normally P: γ x + β y + δ = z, [6] but $\beta = 0$ since P//y′y).

You can easily verify both on a new GeoGebra sheet too. Eliminating z from (2) and (3) we get:

$$\alpha^2x^2+\alpha^2y^2=(\gamma x+\delta)^2 \Leftrightarrow \alpha^2x^2+\alpha^2y^2=\gamma^2x^2+2\gamma\delta x+\delta^2 \Leftrightarrow \left(\alpha^2-\gamma^2\right)x^2+\alpha^2y^2-2\gamma\delta x=\delta^2 \tag{8}$$

1) If $\gamma = 0$, (8)

$$\Leftrightarrow \alpha^2x^2+\alpha^2y^2=\delta^2 \Leftrightarrow x^2+y^2=\frac{\delta^2}{\alpha^2}, \tag{9}$$

which as independent of z in 2 dimensions is an equation of a Circle having radius $r=\dfrac{|\delta|}{\alpha}$ end centre K on O(0,0) on the Oxy plane, but in 3 dimensions points the vertical circular Cylinder $x^2+y^2+0z=\dfrac{\delta^2}{\alpha^2}$ [5] (Fig. 13).

And that is because in the rectangular triangle OK_1A_1, $tan\omega'=\dfrac{r}{|\delta|}\Leftrightarrow cot\omega=\dfrac{r}{|\delta|}\Leftrightarrow\dfrac{1}{tan\omega}=\dfrac{r}{|\delta|}\Leftrightarrow\dfrac{1}{\alpha}=\dfrac{r}{|\delta|}\Leftrightarrow r=\dfrac{|\delta|}{\alpha}$

1) If $0<\gamma<\alpha$, and $\beta=\sqrt{\alpha^2-\gamma^2}\Leftrightarrow\beta^2=\alpha^2-\gamma^2$,

$$(8)\Leftrightarrow\beta^2x^2-2\frac{\gamma\delta}{\beta}\beta x+\left(\frac{\gamma\delta}{\beta}\right)^2+\alpha^2y^2=\delta^2+\left(\frac{\gamma\delta}{\beta}\right)^2\Leftrightarrow\left(\beta x-\frac{\gamma\delta}{\beta}\right)^2$$

$$+\alpha^2y^2=\frac{\delta^2\beta^2+\gamma^2\delta^2}{\beta^2}\Leftrightarrow\beta^2\left(x-\frac{\gamma\delta}{\beta^2}\right)^2+\alpha^2y^2=\frac{\delta^2(\alpha^2-\gamma^2)+\gamma^2\delta^2}{\beta^2}\Leftrightarrow\beta^2\left(x-\frac{\gamma\delta}{\beta^2}\right)^2+\alpha^2y^2$$

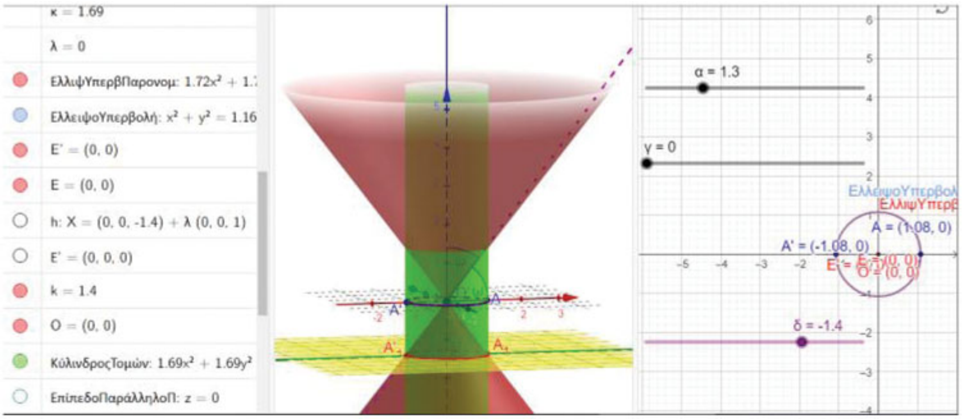


Fig. 13. Plane P//Oxy cutting circle & circular solution cylinder.

$$= \frac{\delta^2\alpha^2}{\beta^2} \Leftrightarrow \frac{\left(x-\frac{\delta}{\beta^2}\gamma\right)^2}{\frac{\delta^2\alpha^2}{\beta^4}} + \frac{\alpha^2y^2}{\frac{\delta^2\alpha^2}{\beta^2}} = 1 \Leftrightarrow \frac{\left(x-\frac{\delta}{\beta^2}\gamma\right)^2}{\frac{\delta^2\alpha^2}{\beta^4}} + \frac{y^2}{\frac{\delta^2}{\beta^2}} = 1 \Leftrightarrow \frac{\left(x-\frac{\delta}{\beta^2}\gamma\right)^2}{\frac{\delta^2}{\beta^4}\alpha^2} + \frac{y^2}{\frac{\delta^2}{\beta^4}\beta^2} = 1 \quad (10)$$

$$\Leftrightarrow \frac{\left(x-\frac{\delta}{\alpha^2-\gamma^2}\gamma\right)^2}{\frac{\delta^2}{(\alpha^2-\gamma^2)^2}\alpha^2} + \frac{y^2}{\frac{\delta^2}{(\alpha^2-\gamma^2)^2}\left(\alpha^2-\gamma^2\right)} = 1$$

which in 2 dimensions is an equation of a displaced [7] Ellipse independent of z, so on a horizontal plane, having the form; $\frac{X^2}{a^2}+\frac{\Upsilon^2}{b^2}=1$, with major axis at x, centre $K\left(\frac{\delta}{\alpha^2-\gamma^2}\gamma,0\right)$, which lies on the left of O(0,0), semi-major axis $\mathbf{a}=\frac{|\delta|}{\beta^2}\alpha$, semi-minor axis $\mathbf{b}=\frac{|\delta|}{\beta^2}\beta$, focal distance from K, $\boldsymbol{c}=\sqrt{a^2-b^2}$ $=\sqrt{\frac{\delta^2\alpha^2}{\beta^4}-\frac{\delta^2}{\beta^4}\beta^2}=\sqrt{\frac{\delta^2\alpha^2-\delta^2\beta^2}{\beta^4}}=\frac{\sqrt{\delta^2\gamma^2}}{\beta^2}=\frac{-\delta}{\alpha^2-\gamma^2}\gamma$, eccentricity $\varepsilon=\frac{\boldsymbol{c}}{\boldsymbol{a}}=\frac{\frac{-\delta}{\beta^2}\boldsymbol{\gamma}}{\frac{-\delta}{\beta^2}\boldsymbol{\alpha}}=\frac{\boldsymbol{\gamma}}{\boldsymbol{\alpha}}$, but in 3 dimensions indicates the vertical elliptic Cylinder (Fig. 14): $\frac{x^2}{a^2}+\frac{y^2}{b^2}+0z=1$ [5].

Then the Focus points of the Ellipse are located on the x-axis on either side of $K\left(\frac{\delta}{\beta^2}\gamma,0\right)$ at a distance $\mathrm{c}=\frac{|\delta|}{\beta^2}\gamma$ from it, therefore, the Focus points are $\mathrm{E}'\left(\frac{\delta}{\beta^2}\gamma+\frac{\delta}{\beta^2}\gamma,0\right)$ and $E\left(\frac{\delta}{\beta^2}\gamma-\frac{\delta}{\beta^2}\gamma,0\right)$, so is $\mathrm{E}'\left(2\frac{\delta}{\alpha^2-\gamma^2}\gamma,0\right)$ while is E(0,0), so E' is on the left of O(0,0), and E is on O(0,0) when $\delta<0$.

So, we have shown that indeed the z′z axis intersects this Ellipse at one of its Focus points, as we also experimentally found by using GeoGebra.

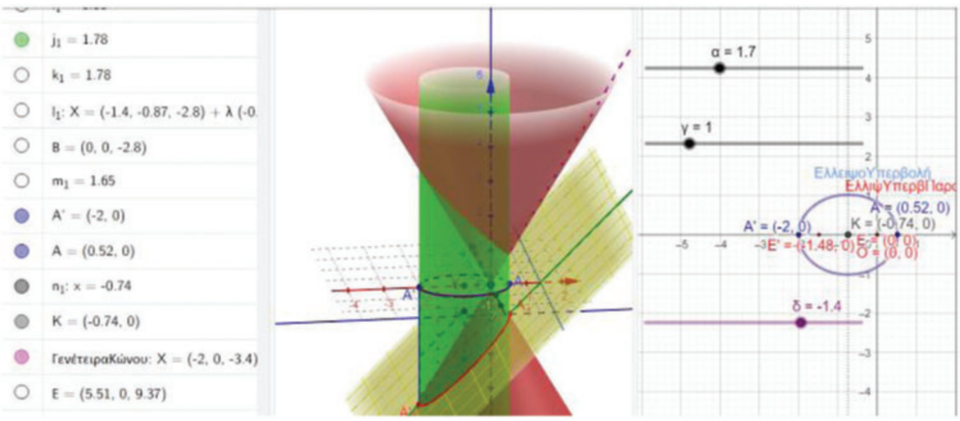


Fig. 14. Plane P cutting ellipse & elliptic solution cylinder.

2) If $\gamma > \alpha$, and $\beta = \sqrt{\gamma^2-\alpha^2} \Leftrightarrow \beta^2=\gamma^2-\alpha^2 \Leftrightarrow -\beta^2=\alpha^2-\gamma^2$, so

$$(8) \Leftrightarrow -\beta^2x^2-2\frac{\gamma\delta}{\beta}\beta x+\alpha^2y^2=\delta^2 \Leftrightarrow \beta^2x^2+2\frac{\gamma\delta}{\beta}\beta x+\left(\frac{\gamma\delta}{\beta}\right)^2-\alpha^2y^2=-\delta^2+\left(\frac{\gamma\delta}{\beta}\right)^2$$

$$\Leftrightarrow \left(\beta x+\frac{\gamma\delta}{\beta}\right)^2-\alpha^2y^2=\frac{\delta^2\gamma^2-\delta^2\beta^2}{\beta^2} \Leftrightarrow \beta^2\left(x+\frac{\gamma\delta}{\beta^2}\right)^2-\alpha^2y^2=\frac{\delta^2\gamma^2-\delta^2(\gamma^2-\alpha^2)}{\beta^2} \Leftrightarrow$$

$$\beta^2\left(x+\frac{\gamma\delta}{\beta^2}\right)^2-\alpha^2y^2=\frac{\delta^2\alpha^2}{\beta^2} \Leftrightarrow \frac{\left(x+\frac{\gamma\delta}{\beta^2}\right)^2}{\frac{\delta^2\alpha^2}{\beta^4}}-\frac{\alpha^2y^2}{\frac{\delta^2\alpha^2}{\beta^2}}=1 \Leftrightarrow \frac{\left(x+\frac{\gamma\delta}{\beta^2}\right)^2}{\frac{\delta^2\alpha^2}{\beta^4}}-\frac{y^2}{\frac{\delta^2}{\beta^2}} \tag{11}$$

$$=1 \Leftrightarrow \frac{\left(x+\frac{\delta}{\beta^2}\gamma\right)^2}{\frac{\delta^2}{\beta^4}\alpha^2}-\frac{y^2}{\frac{\delta^2}{\beta^4}\beta^2}=1$$

$$\Leftrightarrow \frac{\left(x+\frac{\delta}{\gamma^2-\alpha^2}\gamma\right)^2}{\frac{\delta^2}{(\gamma^2-\alpha^2)^2}\alpha^2}-\frac{y^2}{\frac{\delta^2}{(\gamma^2-\alpha^2)^2}(\gamma^2-\alpha^2)}=1$$

which in 2 dimensions is an equation of a displaced [7] Hyperbola independent of z, so on an horizontal plane Oxy, in form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, *with major axis at* $x'x$, *and centre* $K\left(\frac{-\delta}{\beta^2}\gamma, 0\right)$, *which lies on the right of* $O(0,0)$, *semi−major axis* $\boldsymbol{a}=\frac{\delta}{\gamma^2-\alpha^2}\alpha$, *semi−minor axis* $\boldsymbol{b}=\frac{\delta}{\gamma^2-\alpha^2}\beta$, *Focal length from K*; $\boldsymbol{c}=\sqrt{a^2+b^2}=\sqrt{\frac{\delta^2\alpha^2}{\beta^4}+\frac{\delta^2}{\beta^2}}=\sqrt{\frac{\delta^2\alpha^2+\delta^2\beta^2}{\beta^4}}=\frac{\sqrt{\delta^2\gamma^2}}{\beta^2}=\frac{\delta}{\gamma^2-\alpha^2}\gamma$, *and Eccentricity* $\varepsilon=\frac{\boldsymbol{c}}{\boldsymbol{a}}=\frac{\frac{\delta}{\beta^2}\boldsymbol{\gamma}}{\frac{\delta}{\beta^2}\boldsymbol{\alpha}}=\frac{\boldsymbol{\gamma}}{\boldsymbol{\alpha}}$. *Then the focus points of the Hyperbola are located on the x-axis on either side of* $K\left(\frac{-\delta}{\beta^2}\gamma, 0\right)$ *and at a distance*

$\gamma=\frac{-\delta}{\beta^2}\gamma$ *from it, so* $E'\left(\frac{-\delta}{\beta^2}\gamma+\frac{-\delta}{\beta^2}\gamma, 0\right)$ *and* $E\left(\frac{-\delta}{\beta^2}\gamma-\frac{-\delta}{\beta^2}\gamma, 0\right)$, *so* $E(0,0)$ *lies on* $O(0,0)$ *and* $E'\left(-2\frac{\delta}{\beta^2}, 0\right)$ *is on the right of the points O, K and E, by* $\delta<0$ *and on the left side when* $\delta<0$. Though in 3 dimensions (Fig. 15) (11) shows the [5] vertical hyperbolic Cylinder:

$\frac{\boldsymbol{x^2}}{\boldsymbol{a^2}}-\frac{\boldsymbol{y^2}}{\boldsymbol{b^2}}+0z=1.$

So, we proved again that the z′z axis indeed intersects this Hyperbola at one of its Focus points, as we also experimentally found with GeoGebra.

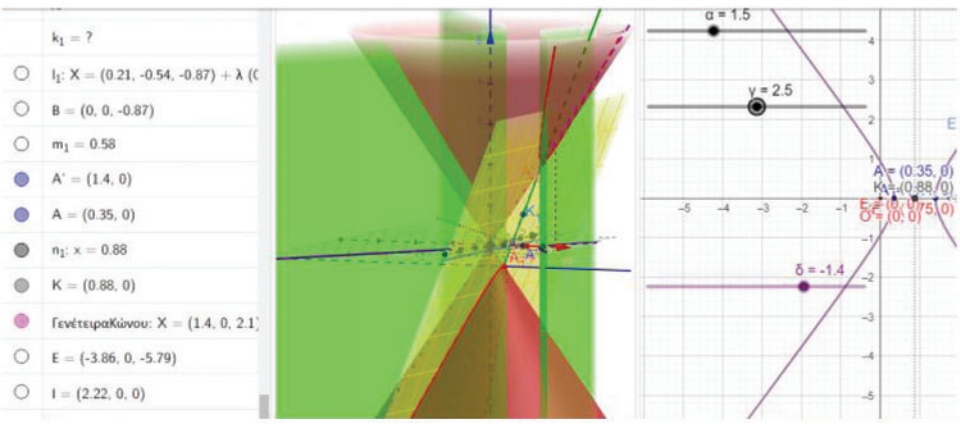


Fig. 15. Plane P cutting hyperbola on the conic surface CS.

3) If γ = α, then

$$(8) \Leftrightarrow \alpha^2y^2 - 2\gamma\delta x = \delta^2 \Leftrightarrow \alpha^2y^2 = 2\alpha\delta x + \delta^2 \Leftrightarrow y^2 = 2\frac{\alpha\delta}{\alpha^2}x + \frac{\delta^2}{\alpha^2} \Leftrightarrow y^2 = 2\frac{\delta}{\alpha}x + \frac{2\delta^2}{2\alpha^2} \Leftrightarrow y^2$$

$$= 2\frac{\delta}{\alpha}\left(x + \frac{1}{2}\frac{\delta}{\alpha}\right) \Leftrightarrow y^2 = 2\frac{\delta}{\alpha}\left(x + \frac{\frac{\delta}{\alpha}}{2}\right) \tag{12}$$

$$\Leftrightarrow y^2 = 2p\left(x + \frac{p}{2}\right)$$

which in 2 dimensions is an equation of a displaced [7] Parabola C having the form $y^2 = 2px$, independent of z, so it is on a horizontal plane Oxy, having x′x as axis of symmetry its vertex is the point $A\left(\frac{-\delta/\alpha}{2}, 0\right)$ so is $A(-p/2, 0)$ that lies on the right of O(0,0), $p = \delta/\alpha < 0$, so C is on the left of the point A for $\delta < 0$ and at the opposite side if $\delta > 0$.

Then the coordinate line d: $X = -\frac{p}{2} \Leftrightarrow x + \frac{\delta/\alpha}{2} = \frac{-\delta/\alpha}{2} \Leftrightarrow x = -\delta/2\alpha - \delta/2\alpha \Leftrightarrow x = -2\delta/2\alpha \Leftrightarrow x = -\delta/\alpha \Leftrightarrow x = -p$.

So, the Focus point $E\left(\boldsymbol{x_A} + \frac{\boldsymbol{p}}{\boldsymbol{2}}\right) = E\left(\frac{-\frac{\delta}{\alpha}}{\boldsymbol{2}} + \frac{\delta\alpha}{2}, 0\right) = E\,(0, 0)$ and we have shown again that axis z′z indeed intersects this Parabola at its Focus point E, as we had also experimentally found by using GeoGebra. Also, in 3 dimensions we take the vertical parabolic Cylinder $y^2 = 2px + 0z$ [5] (Fig. 16).

Then, when $\delta = 0$ the (yellow) plane P tutches (Fig. 17) both of the upper and down part of the double conic surface along a Generator line of the Conic surface and (12) turns to y = 0.

In this case in 2 dimensions we have as solution on the left, the blue projection of the Generator line to Oxy plane that is axis x′x: y = 0 and also on the right graph which is the degenerate parabola we have mentioned before. Also in 3 dimensions, we have the solution of the vertical plane Oxz: y = 0 as a degenerated vertical parabolic Cylinder. And this is of course the case where in the previous paragraph (α) we have α = γ and x′x axis appears between the Ellipse and Hyperbola, which as we said there, is the case of the degenerated parabola.

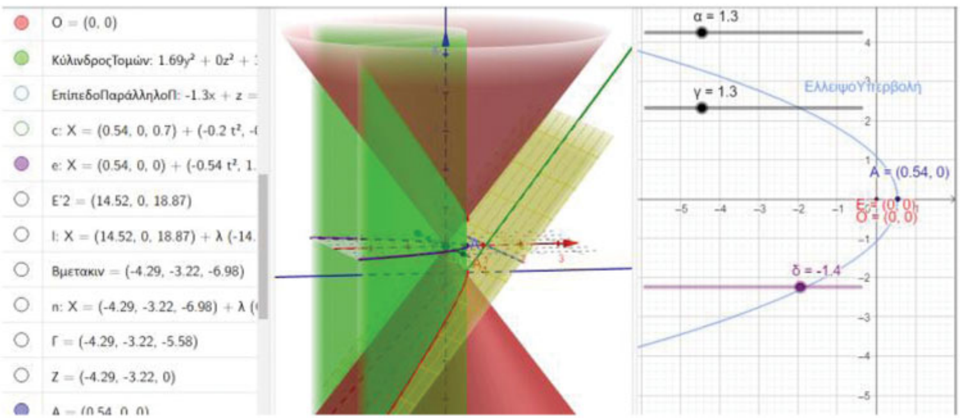


Fig. 16. Plane P cutting Parabola & Parabolic solution Cylinder.

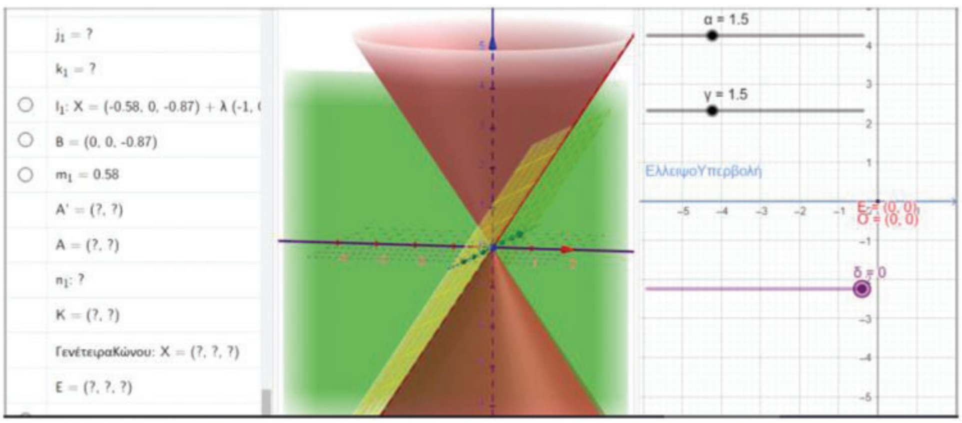


Fig. 17. Plane P touching ellipse & planeas solution “Cylinder”.

3.3. Summarizing

The above-written (8) is the one depicted on the right of the GeoGebra sheet, and as we have already shown above, in any case it is an equation of a Conic Section independent of z, so on a horizontal plane, i.e., the projection of the common points of the double conic surface with the intersecting (yellow) plane on the (grey) plane Oxy: z = 0, where z = 0 and that is why z does not appear in the equation.

In all three cases, the z′z axis intersects each of the formed conic sections at one of its Focus points.

For the equation of the real conic section in 3dimensioned space on the (yellow) plane (which will also include the z variable), and for its equation on the intersecting (yellow) plane (without the z variable), which needs rotation theory in 3D space, we will write here.

For the first one, that is the 3 dimensional equation of the inclined conic section this it cannot be defined, as all 3 dimensional equations define 3d surfaces. So, this inclined conic section can be defined as the solution of a system of two 3 dimensioned equations; here, the defining system of this 3D curve can be firstly the equation of the yellow plane P: γ x + δ = z, mentioned above as (7) and secondly either the initial equation of the double conic surface CS: $\alpha^2x^2 + \alpha^2y^2 = z^2$, mentioned above as (6), or by the general (8) or one of the equivalent to (8) equations; (9) of the Circle, or (10) of the Ellipse, or (11) of the Hyperbola, or (12) of the Parabola, by changing the values of a and c.

For the second one, that is the 2 dimensional equation of the inclined conic section on a Cartesian coordinates system $E_1\chi\Psi$ of two perpendicular axes on the intersecting (yellow) plane P, the axis $\chi'\chi$ defined as the intersection of the vertical plane y = 0 (that includes z′z axis) to plane P, and as the second axis ψ′ψ be the parallel to axis y′y straight line including the point E_1, that is of course the line χ = 0 on this coordinates system $E_1\chi\Psi$.

So, (in Fig. 18):

1. In the case of Ellipse-Hyperbola we have seen that we have a common equation, the equation;

C: $\dfrac{\left(x-\frac{\delta}{\alpha^2-\gamma^2}\gamma\right)^2}{\frac{\delta^2}{(\alpha^2-\gamma^2)^2}\alpha^2} + \dfrac{y^2}{\frac{\delta^2}{(\alpha^2-\gamma^2)^2}\left(\alpha^2-\gamma^2\right)} = 1$, we found above and $\kappa = \dfrac{\delta}{\beta^2} \Leftrightarrow \dfrac{(x-\kappa\gamma)^2}{\kappa^2\alpha^2} + \dfrac{y^2}{\kappa^2\left(\alpha^2-\gamma^2\right)} = 1 \Leftrightarrow$

$\dfrac{(x+c)^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where a = $\kappa\alpha$, b = $\kappa\beta$ and c = $\sqrt{a^2+b^2} = \kappa\gamma$.

We will find a_1, b_1 και c_1 of the 2-dimmensional inclined conic section C_1: $\dfrac{(\chi+c_1)^2}{a_1^2} + \dfrac{\psi^2}{b_1^2} = 1$, coresponding of course to the great semi-axis, the small semi-axis and the semi-distance of the two Focus points of the inclined conic section C_1.

So, if C_2 is the parallel to z′z displacement of C_1, $|\delta|$ units up so that its Focus point E_2 comes to O. Then in the rectancular triancle $OA'A_2'$ and $KK_2//A'A_2'$ from Thales theorem [8]

$\Rightarrow \dfrac{OK}{OK_2} = \dfrac{AK}{A_2K_2} = cos\theta \Rightarrow \dfrac{c}{c_2} = \dfrac{a}{a_2} = cos\theta \Rightarrow \dfrac{c}{c_1} = \dfrac{a}{a_1} = cos\theta \Rightarrow c_1 = \dfrac{c}{cos\theta}$ and $a_1 = \dfrac{a}{cos\theta}$.

and as $0° < \theta < 90°$, and $cos^2\theta + sin^2\theta = 1$ [7] $\Rightarrow 1 + tan^2\theta = \dfrac{1}{cos^2\theta} \Rightarrow \dfrac{1}{cos^2\theta} = 1 + \gamma^2 \Rightarrow 1/cos\theta = \sqrt{1+\gamma^2} = \lambda$,

so $c_1 = \lambda c, a_1 = \lambda a$ and $b_1 = \lambda b$.

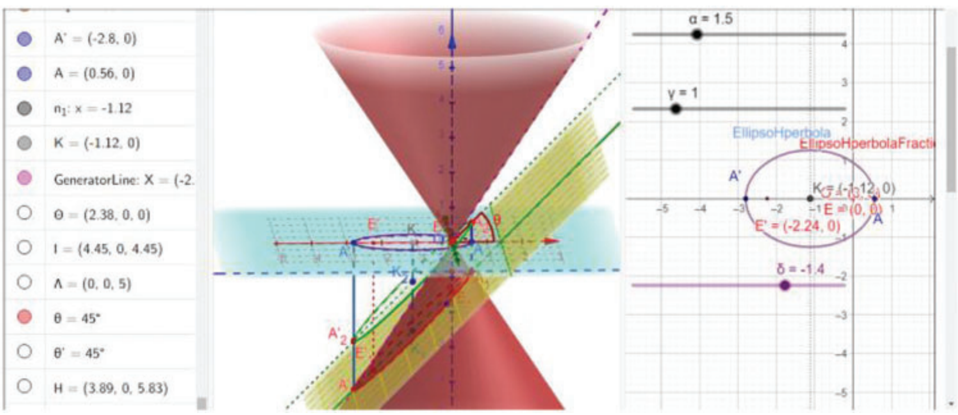


Fig. 18. Calculating Conic Section C_1 on Plane p.

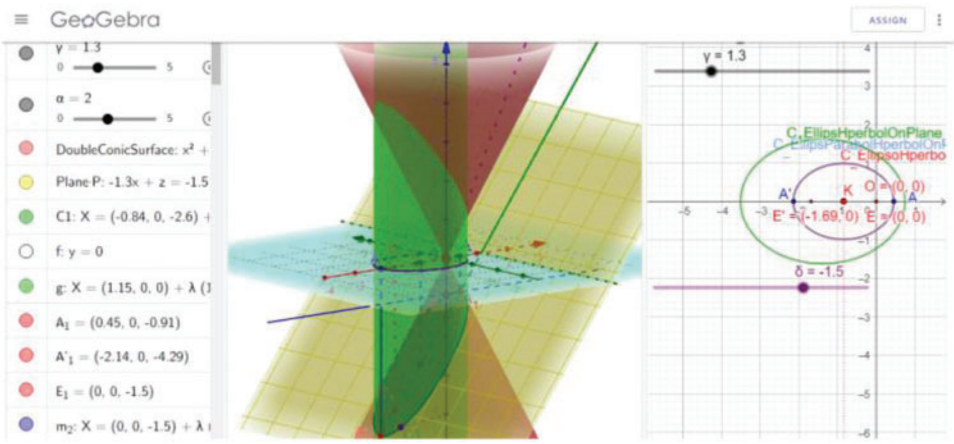


Fig. 19. C_1 appears right by ticking the green C_1 button on the left.

So, the equation of the inclined Ellipse C_1 at coordinates of the coordinates system $E_2\chi\Psi$ on the plane P is:

$$C1: \frac{\left(\chi+\frac{|\delta|}{\beta^2}(1+\gamma^2)\gamma\right)^2}{\frac{\delta^2}{\beta^4}(1+\gamma^2)\alpha^2}+\frac{\psi^2}{\frac{\delta^2}{\beta^2}(1+\gamma^2)\beta^2}=1 \Longleftrightarrow C1: \frac{(\chi+\kappa\lambda\gamma)^2}{(\kappa\lambda\alpha)^2}+\frac{\psi^2}{(\kappa\lambda\beta)^2}=1 \tag{13}$$

having Eccentricity: $\varepsilon_1 = \dfrac{c_1}{a_1} = \dfrac{c}{a} = \dfrac{\gamma}{\alpha}$.

2) In the same way at the case of having the Parabola C numbered above as (12), for the inclined Parabola C_1, at coordinates of $O_1\chi\Psi$ system on the plane P we can find respectively that $c_1 = \lambda c$ and $p_1 = \lambda p$, where $\lambda = \sqrt{1+\gamma^2}$, $c = \dfrac{-\delta/\alpha}{2}$ and $p = \delta/\alpha$. So, having $\chi'\chi$ as axis of symmetry, it has vertex the point $A_1\left(-\lambda\dfrac{\delta/\alpha}{2}, 0\right)$, that lies on the right of $O_1(0,0)$, $p_1 = \lambda\dfrac{\delta}{\alpha} < 0$, so C_1 is on the left of A_1, if $\delta < 0$ and just the oposite if $\delta > 0$.

Then the coordinate line d_1: $X = -\dfrac{p_1}{2} \Leftrightarrow \chi + \lambda\dfrac{\frac{\delta}{\alpha}}{2} = \boldsymbol{\lambda}\dfrac{-\frac{\delta}{\alpha}}{2} \Leftrightarrow \chi = -\lambda\dfrac{\delta}{\alpha} \Leftrightarrow \chi = -\lambda p$

The Focus pointis $E_1\left(\boldsymbol{\chi_{A_1}}+\boldsymbol{\lambda}\dfrac{\boldsymbol{p}}{\boldsymbol{2}},\boldsymbol{0}\right) = E_1\left(-\lambda\dfrac{\boldsymbol{\delta/\alpha}}{\boldsymbol{2}}+\lambda\dfrac{\delta/\alpha}{2},0\right) = E_1\,(0,0)$. So, E_1 is located on $O_1(0,0)$, and

$$C_1: \psi^2 = 2\sqrt{1+\gamma^2}\frac{\delta}{\alpha}\left(\chi+\sqrt{1+\gamma^2}\frac{\frac{\delta}{\alpha}}{2}\right) \tag{14}$$

$$\Leftrightarrow C1: \psi^2 = 2\lambda p\left(\chi+\lambda\frac{p}{2}\right) \tag{15}$$

In the case of C being a Circle there is no inclined conic section as C_1 is horizontal and identical to C, that is a case it has been already studied above.

All these equations of the various types of C_1 that we have found before can easily been tested by activating $C_1$by ticking on it to the left algebra window of GeoGebra. Then C_1 curve apears in green on the right graph resembling to C as they share the same eccentrisity but bigger than C (as we can see in Fig. 19).

4. Conclusions

So, as we have proven above and observed by GeoGebra, C and C_1 in their coordinate systems are each other displacements:

1) As δ is where the plane P intersects axis z′z, α = tanω is the slope and ω is the inclination of the Generator line of the duble conic surface and γ = tanθ is the slope and θ the inclination of the intersecting plane P to the horisontal plane Oxy, the position of C is at $\dfrac{-\delta}{\alpha^2-\gamma^2}$ units to the left

(in Ellipse) or the equivelent $\frac{-\delta}{\gamma^2-\alpha^2}$ units to the right (in Hperbola), of the symmetric around O(0,0) Conic Sections C_0, at the beginning of this work, having equations:

C_0: $(a^2 - c^2)\, x^2 + a^2y^2 = a^2\, (a^2 - c^2)$, mentioned above by the (2), where in C_0 $is\, c = \frac{|\delta|}{|\alpha^2-\gamma^2|}\gamma$, a $= \frac{|\delta|}{|\alpha^2-\gamma^2|}\alpha$.

The Focus points of C_0 are $E_0\left(\frac{\delta}{\alpha^2-\gamma^2}-\frac{\delta}{\alpha^2-\gamma^2},0\right)=O\,(0,0)$ and $E_0'\left(\frac{\delta}{\alpha^2-\gamma^2}+\frac{\delta}{\alpha^2-\gamma^2},0\right)\Rightarrow$ $E_0'\left(\frac{2\delta}{\alpha^2-\gamma^2},0\right)$ is located lefter than O(0,0) in Ellipse for $\alpha > \gamma$, and righter than O(0,0) in Hyperbola for $\alpha < \gamma$, in both cases for $\delta < 0$ and just oposite when $\delta > 0$.

2) C_1 is the displacement of C_0 at $\frac{-\delta}{\alpha^2-\gamma^2}\sqrt{1+\gamma^2}$ units to the left (in Ellipse) or the equivelent $\frac{-\delta}{\gamma^2-\alpha^2}\sqrt{1+\gamma^2}$ units to the right (in Hperbola) where in C_0, $c = \frac{-\delta}{\alpha^2-\gamma^2}\gamma\sqrt{1+\gamma^2}$ and a $= \frac{-\delta}{\alpha^2-\gamma^2}\alpha\sqrt{1+\gamma^2}$.

The Focus points of C_1 are $E_1\left(\frac{\delta}{\alpha^2-\gamma^2}\sqrt{1+\gamma^2}-\frac{\delta}{\alpha^2-\gamma^2}\sqrt{1+\gamma^2},0\right) = O\,(0,0)$ and $E_1'\left(\frac{\delta}{\alpha^2-\gamma^2}\sqrt{1+\gamma^2}+\frac{\delta}{\alpha^2-\gamma^2}\sqrt{1+\gamma^2},0\right) = \left(\frac{2\delta\sqrt{1+\gamma^2}}{\alpha^2-\gamma^2},0\right)$ is located lefter than O(0,0) in Ellipse as $\alpha > \gamma$, and righter than O(0,0) in Hyperbola as $\alpha < \gamma$ with $\delta < 0$, and just the oposite if $\delta > 0$.

3) So C is the projection of C_1 on the horisontal plane Oxy and the symmetrical conic sections C_0 of (1) is the displacement of C along axis x′x at $\frac{-\delta}{\alpha^2-\gamma^2}$ units to the right for Helipses or equivelantly at $\frac{-\delta}{\gamma^2-\alpha^2}$ units to the left for Hyperbolas, where a and c are difined as just above in C or C_1 depending of which equation of them we know.

i.e., from C_1 to C and then to C_0, the elements of C in terms of the elements of C_1 are:
$a = a_1/\sqrt{1+\gamma^2}$, $c = c_1/\sqrt{1+\gamma^2}$ and $b = b_1/\sqrt{1+\gamma^2}$

4) Parabolas are not symmetrical around a point normally, only the disentigrated Parabola ψ = 0 on Plane P, that is the Generator line of the conic surface, it is Symmetrical around O. This C_1: $\psi = 0$ descends from the equation of C_1:$\psi^2 = 2\sqrt{1+\gamma^2}\frac{\delta}{\alpha}\left(\chi+\sqrt{1+\gamma^2}\frac{\delta/\alpha}{2}\right)$, mentioned above as (14), for $\delta = 0$ and it has as projection the C: y = 0 that is axis x′x descending from C: $y^2 = 2\frac{\delta}{\alpha}\left(x+\frac{\delta/\alpha}{2}\right)$ for $\delta = 0$. So, the only symmetrical parabola C_0 is axis x′x that comes also from (2) for a = c and $\delta = 0$.

Recall that a two-dimensional curve with equation f(x, y) = 0 has center of symmetry O(0,0) if and only if also f(−x, −y) = 0.

Obviously all the normally teached equations; $x^2+y^2=r^2$ of Circles $\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1$ of Ellipses and the equivalent equation $\frac{x^2}{a^2}-\frac{y^2}{c^2-a^2}=1$ of Hyperbolas can be writen as $(-x)^2+(-y)^2=r^2$, $\frac{(-x)^2}{a^2}+\frac{(-y)^2}{a^2-c^2}=1$ and $\frac{(-x)^2}{a^2}-\frac{(-y)^2}{c^2-a^2}=1$, so all these curves are symmetrical arount O(0,0).

As a matter of fact Parabola is not usually symmetric around O(0,0), as $y^2=2px\,(15) \not\Leftrightarrow (-y)^2 = 2p\,(-x)\,for\,p\neq 0$.

Nevertheless if p = 0, (15) $\Leftrightarrow y^2=0 \Leftrightarrow y=0$, that is axis x′x, that is the disentigrated parabola we have already mentioned above. This is a symmetric arount O(0,0) curve, as $y=0 \Leftrightarrow -y=0$ and for any $x\in x'x \Leftrightarrow -x \in x'x$.

5) For $\gamma = 0$ when P//Oxy, $c = \frac{|\delta|}{|\alpha^2-\gamma^2|}\sqrt{1+\gamma^2}\gamma = 0$, and the above initial (2) turns to $a^2\, x^2 + a^2\, y^2 = a^4$

6) $\Leftrightarrow x^2 + y^2 = a^2$, that is a Circle of radius a= $\dfrac{|\delta|}{|\alpha^2 - \gamma^2|}\sqrt{1+\gamma^2} \Longrightarrow \alpha = |\delta|/\alpha$ (for $\gamma = 0$) as we have already calculated above by Geometry (in (9) on section B(i)). So (2) can be tranformed also to a Circle, that is of course also a symmetric curve around the point O(0,0).

Therefore the (2) of Ellipso–Yperbola is finally the common equation of all symmetrical Conic Section around O(0,0) that is the Circle, the Ellipse, Hyperbola and the disentigrated Parabola x′x, in their normally teached form.

7) All cases of C and C_1 are intersected by axis z′z on one of their Focus points, here on E and E_1.
8) All Conic Sections of Ellipse or Hyperbola C and C_1 share the same Eccentrity $\varepsilon = \gamma/\alpha$, where α is the slope of the Generator line of the Conic Surface CS and γ is the slope of the plane P towards plane Oxy.
9) We have also proven (what we have also observed by GeoGebra) that during the transformation of Ellipse to Hyperbola the central point K and the left Focus point E′ they both move away from the Focus point E towards $-\infty$, while the Conic Section is transforming initially from Ellipse to Parabola and then to Hyperbola the Focus point E′ and the central point K both appear as coming from $+\infty$ on the right side of the Focus point E, as an indication of a circularity of the two and three-dimensional space, but this is a Topologic matter that needs to be examined separately, as the double conical surface acts as a "Klein's bottle", i.e. a three-dimensional surface, where the inner Euclidean half-space communicates with the outer one through the connection point of the upper and lower Cone in the Minkowski 4-dimensional spacetime, while the double conical surface is produced by rotating a generating straight line around the z′z axis by varying the value of time (See an ant walking along the Generator straight line that produces the Conic Surface, by time passing at the link : https://www.geogebra.org/m/pv47kjsk).

Conflict of Interest

Author declares no conflict of interest.

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