Suzuki-Type of Common Fixed Point Theorems in S-Fuzzy Metric Spaces

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Abstract — In this paper, by using of Suzuki-type approach [Suzuki, T., A generalized Banach contraction principle that characterizes metric completeness, Proc. Amer. Math. Soc., 136, 1861–1869, 2008.] we prove new type of Suzuki-type fixed point theorem for non-Archimedean S-fuzzy metric spaces which is generalization of Suzuki-Type fixed point results in S-metric spaces.

Key words — Fixed point, Suzuki-Contraction, S-fuzzy metric space.

I. INTRODUCTION

In 1965, the concept of the fuzzy set was initially investigated by Zadeh [13]. Then in 1975, Kramosil and Michalek [4] introduced the fuzzy metric space as a generalization of a metric space. In 1994, George and Veeramani [2] modified the notion of fuzzy metric spaces by using continuous t-norms. Fixed point theorems for contractive mappings in metric spaces have been studied by many authors (see [1], [5], [12]). The Banach contraction principle is the most celebrated fixed point theorem and has been generalized in various directions. In 2008 Suzuki [6] introduced an interesting generalization of Banach contraction principle. Recently, Sedghi, Shobe and Aliouche [8] have defined the concept of S-metric space as a generalization of a metric space, (see [3], [7], [9]-[11]) and proved some fixed point results. In this paper, we introduce the new contractive condition in the frame work of non-Archimedean S-fuzzy metric spaces. We also prove the corresponding coincidence fixed point theorem for two mappings in this framework.

II. PRELIMINARIES

A. Definition 2.1

A binary operation $*: [0, 1] \times [0, 1] \to [0, 1]$ is a continuous $t-$ norm if it satisfies the following conditions:

1. $*$ is associative and commutative;
2. $*$ is continuous;
3. $a * 1 = a$, for all $a \in [0, 1]$;
4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$.

Examples for continuous $t-$ norm are $a \times b = \min \{a, b\}$ and $a \times b = ab$.

B. Definition 2.2

A 3 – tuple $(X, S, *)$ is called S-fuzzy metric space if $X$ is an arbitrary non – empty set, $*$ is a continuous $t-$ norm, and $S$ is a fuzzy set on $X^3 \times [0, \infty)$ satisfying the following conditions. For each $x, y, z, a \in X$ and $r, s, t > 0$:

1. $S(x, y, z, t) > 0$;
2. $S(x, y, z, t) = 1$ if $x = y = z$;
3. $S(x, y, z, t) = S(p(x, y, z), t)$, where $p$ is a permutation;
4. $S(x, y, w, r) * S(x, w, z, s) * S(w, y, z, t) \leq S(x, y, z, r + s + t)$, (Tetrahedral inequality) .
5. $S(x, y, z, .): [0, \infty) \to [0, 1]$ is continuous.

C. Example 2.3

Let $X = [0, 1]$ with a usual metric. Define

$$S(x, y, z, t) = \min \{M(x, y, t), M(y, z, t), M(x, z, t)\}$$

where $M(x, y, t) = \frac{t}{t + d(x, y)}$ and $d(x, y) = |x - y|$ for all $x, y \in X$. Then $(X, S, *)$ is called S–fuzzy metric space.

D. Definition 2.4

A 3 – tuple $(X, S, *)$ is called non-Archimedean S-fuzzy metric space if $X$ is an arbitrary non – empty set, $*$ is a continuous $t-$ norm, and $S$ is a fuzzy set on $X^3 \times [0, \infty)$ satisfying the following conditions. For each $x, y, z, w \in X$ and $r, t, u > 0$:

1. $S(x, y, z, t) > 0$;
2. $S(x, y, z, t) = 1$ if $x = y = z$;
3. $S(x, y, z, t) = S(p(x, y, z), t)$, where $p$ is a permutation;
4. $S(x, y, w, r) * S(x, w, z, s) * S(w, y, z, t)$

$$\leq S(x, y, z, r \lor t \lor u),$$

where $r \lor t \lor u = \max \{r, t, u\}$, (Tetrahedral inequality) .
5. $S(x, y, z, .): [0, \infty) \to [0, 1]$ is continuous.

E. Definition 2.5

Let $(X, S, *)$ be a S-fuzzy Metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then:

1. A sequence $\{x_n\}$ is said to be convergent to $x$ if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x, \varepsilon) > 1 - \varepsilon$ for all $n \geq n_0$.
2. A sequence $\{x_n\}$ is said to be a Cauchy sequence if for each $\varepsilon > 0$ and there exists $n_0 \in \mathbb{N}$ such that $S(x_m, x_n, \varepsilon) > 1 - \varepsilon$ for all $m, n \geq n_0$.

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3. The $S$-fuzzy metric space is called complete if every Cauchy sequence is convergent.

F. Lemma 2.6

Let $(X, S, \ast)$ be an $S$-fuzzy metric space. If there exists sequence \( \{x_n\} \) and \( \{y_n\} \) such that \( \lim_{n \to \infty} x_n = x \) and \( \lim_{n \to \infty} y_n = y \), then \( \lim_{n \to \infty} S(x_n, x, y_n) = S(x, x, y) \) for all $x, y \in X$.

III. MAIN RESULTS

A. Theorem 3.1

Let $a \ast b = ab$ for all $a, b \in [0, 1]$ and $(X, S, \ast)$ be a complete non-Archimedean $S$-fuzzy metric space. Let $T, R: X \to X$ be two self maps and $\theta: [0, 1) \to (\frac{1}{2}, 1]$ be defined by:

$$
\theta(r) = \begin{cases} 
1, & 0 \leq r \leq \frac{\sqrt{\pi} - 1}{2} \\
\frac{1 - r}{\frac{\sqrt{\pi} - 1}{2}}, & \frac{\sqrt{\pi} - 1}{2} \leq r \leq \frac{1}{\sqrt{2}} \\
\frac{1}{1 + r}, & \frac{1}{\sqrt{2}} \leq r \leq 1 
\end{cases}
$$

(1)

If there exists $r \in (0, 1)$ such that for each $x, y, z \in X, t > 0$ satisfying the condition:

$$
\max(S(x, y, Tx, t), S(x, y, Rx, t))^{\theta(r)} \geq S(x, y, z, t)
$$

implies

$$
\begin{align*}
\{S(Rx, Ry, Rz, t) \ast S(Tx, Ty, Tz, t) \ast & S(Rx, Ry, Tz, t) \ast S(Tx, Ty, Rz, t)\} \\
\geq & S(x, y, z, t)^r 
\end{align*}
$$

(2)

Then there exists a unique common fixed point of $T$ and $R$.

B. Proof

At first, we prove that if $v$ is a fixed point of $T$, then it is also a fixed point of $R$ and vice versa. Hence, let $v$ be a fixed point of $T$, then we will show that $Tv = v$.

Taking $x, y = v$ and $z = Tv$ in (2) and (3), We get:

$$
1 = \max(S(v, v, Tv, t), S(v, v, Rv, t))^{\theta(r)} > S(v, v, Tv, t)
$$

implies

$$
\begin{align*}
\{S(Rv, Rv, RTv, t) \ast S(Tv, Tv, T^2v, t) \ast & S(Rv, Rv, T^2v, t) \ast S(RTv, TTv, t)\} \\
\geq & S(v, v, Tv, t)^r 
\end{align*}
$$

(4)

Thus, $S(Rv, Rv, v, t) = 1$. i.e., $v$ is a fixed point of $R$. Similarly, we can show that if $v$ is a fixed point of $R$, then it is also a fixed point of $T$. Now to prove our theorem, it is enough to prove that $T$ and $R$ has a fixed point.

Putting $x, y = x$ and $z = Tx$ in (2) and (3), we get:

$$
\max\{S(x, x, Tx, t), S(x, x, Rx, t)\}^{\theta(r)} \geq S(x, x, Tx, t)
$$

implies

$$
\{S(Rx, Rx, RTx, t) \ast S(Tx, Tx, T^2x, t) \ast & S(Rx, Rx, T^2x, t) \ast S(RTx, Tx, t)\} \\
\geq & S(x, x, Tx, t)^r 
$$

Hence,

$$
S(Tx, Tx, T^2x, t) \geq S(x, x, Tx, t)^r \text{ for all } x \in X
$$

and

$$
S(Tx, Rx, T^2x, t) \geq S(x, x, Tx, t) \text{ for all } x \in X
$$

Putting $x, y = x$ and $z = Rx$ in (2) and (3). Hence, form

$$
\max(S(x, x, Tx, t), S(x, x, Rx, t))^{\theta(r)} \geq S(x, x, Rx, t)
$$

implies

$$
\begin{align*}
\{S(Rx, Rx, R^2x, t) \ast S(Tx, Ty, TRx, t) \ast & S(Rx, Rx, TRx, t) \ast S(R^2x, Tx, Tx, t)\} \\
\geq & S(x, x, Rx, t)^r 
\end{align*}
$$

Hence,

$$
S(Rx, Rx, TRx, t) \geq S(x, x, Rx, t)^r \text{ for all } x \in X
$$

Let $x_0 \in X$ be arbitrary and form the sequence $\{x_n\}$ by $x_1 = Rx_0$ and $x_{n+1} = Rx_n$ for $n \in \mathbb{N}$ and $\{0\}$. By (6), we have

$$
S(x_{n+1}, x_{n+1}, x_{n+2}) = S(Tx_{n-1}, Tx_{n-1}, RTx_{n-1}, t)
$$

$$
\geq S(x_{n+1}, x_{n+1}, Tx_{n-1}, t)^r
$$

(5)

$$
= S(x_{n+1}, x_{n+1}, x_{n+2}, t)^r
$$

(6)

By (8), (9) we have:

$$
S(x_{n+1}, x_{n+1}, x_{n+2}, t) \geq S(x_{n+1}, x_{n+1}, x_{n+2}, t)^r
$$

(10)

Hence, by induction, we have:

$$
\begin{align*}
S(x_n, x_n, x_{n+1}, t) \geq S(x_n, x_n, x_{n+1}, t)^r & \\
\geq \cdots & \geq S(x_0, x_0, x_1, t)^{r^n}
\end{align*}
$$

(11)

By Tetrahedral inequality in $S$–fuzzy metric space for $m > n$ we have,

$$
S(x_n, x_m, m, t) \geq S(x_n, x_m, x_{m+1}, t) \ast S(x_{m+1}, x_{n+1}, t) \ast \cdots \ast S(x_{n-1}, x_m, t)
$$

$$
\geq S(x_0, x_0, x_1, t)^{r^n} \ast S(x_0, x_0, x_1, t)^{r^{n+1}}
$$
\[ * \cdots * S(x_0,x_0,x_1,t)^{m-1} \]
\[ \geq S(x_0,x_0,x_1,t)^r \cdot S(x_0,x_0,x_1,t)^{m-1} . \]
\[ \cdots \cdot S(x_0,x_0,x_1,t)^{m-1} . \]

As \( 0 < r < 1 \), we have:
\[ S(x_n,x_n,x_m,t) = S(x_0,x_0,x_1,t)^n (1+r+r^2+\cdots+r^{m-n-1}) \]
\[ \geq S(x_0,x_0,x_1,t)^{-1} \rightarrow 1 \text{ as } m \rightarrow \infty. \]

So, we have \( \lim_{n,m \to \infty} S(x_n,x_n,x_m,t) = 1. \)

It follows that \( \{x_n\} \) is a Cauchy sequence. Since \( X \) is Complete \( S \)-fuzzy metric space, there is some \( v \) in \( X \) such that,
\[ \lim_{n \to \infty} Rx_{2n} = \lim_{n \to \infty} x_{2n+1} = v \]
and
\[ \lim_{n \to \infty} Tx_{2n+1} = \lim_{n \to \infty} x_{2n+2} = v. \]

We show that \( v \) is a common fixed point of \( T \) and \( R \). It is enough to prove that \( Tv = v \). Now we consider \( x \in X \) with \( x \neq v. \)
As
\[ \lim_{n \to \infty} S(x_{2n+1},x_{2n+1},Tx_{2n+1},t) = 1 \]
and
\[ \lim_{n \to \infty} S(x_{2n+1},x_{2n+1},x,t) \neq 1, \]
therefore there exists some \( x_{2n+1} \in X \) such that
\[ \max \left\{ S(x_{2n+1},x_{2n+1},Tx_{2n+1},t), S(x_{2n+1},x_{2n+1},Rx_{2n+1},t) \right\} \]
\[ \geq S(x_{2n+1},x_{2n+1},x,t) \]
implies,
\[ S(Rx_{2n+1},Rx_{2n+1},Rx,t) \cdot S(Tx_{2n+1},Tx_{2n+1},Tx,t) \]
\[ \geq S(x_{2n+1},x_{2n+1},x,t)^r. \]

Hence,
\[ S(x_{2n+2},x_{2n+2},Tx,t) = S(Tx_{2n+1},Tx_{2n+1},Tx,t) \]
\[ \geq S(x_{2n+1},x_{2n+1},x,t)^r. \]

Taking limit as \( n \to \infty \), we have:
\[ S(v,v,Tx,t) = \lim_{n \to \infty} S(x_{2n+2},x_{2n+2},Tx,t) \]
\[ \geq \lim_{n \to \infty} S(x_{2n+1},x_{2n+1},x,t)^r \]
\[ = S(v,v,x,t)^r. \]

Therefore, for each \( x \neq v, \)
\[ S(v,v,Tx,t) \geq S(v,v,x,t)^r. \] (12)

Now by induction, we prove that,
\[ S(v,v,T^mv,t) \geq S(v,v,Tv,t) \]
for all \( n \in \mathbb{N}. \) (13)

For \( n = 1, \) the inequality is obvious.
Suppose the inequality (13) is true for \( m \in \mathbb{N} \)
\[ S(v,v,T^mv,t) \geq S(v,v,Tv,t) \]
Now for \( n = m + 1, \) if \( T^mv = v \) then
\[ S(v,v,T^{m+1}v,t) = S(v,v,Tv,t) \]
If \( T^mv \neq v \) then by (12)
\[ S(v,v,T^{m+1}v,t) \geq S(v,v,T^mv,t)^r \]
\[ \geq S(v,v,Tv,t)^r \]
\[ > S(v,v,Tv,t) \] (14)

Hence, the inequality (13) holds for each \( n \in \mathbb{N}. \)
Let us assume that \( TV = v, \) and prove the following inequality by the principle of mathematical induction
\[ S(Tv,Tv,T^n v,t) \geq S(v,v,Tv,t)^r \]
for each \( n \in \mathbb{N} \) (15)
For \( n = 1, \) it is obvious. Further from (5) the inequality (15) holds for \( n = 2. \)
Suppose (15) holds for some \( n > 2, \) then we have:
\[ S(Tv,Tv,v,t) = S(v,v,Tv,t) \]
\[ \geq S(v,v,T^n v,v) \ast 2S(Tv,Tv,T^n v,t) \]
\[ \geq S(v,v,T^n v,v) \ast 2S(v,v,Tv,t)^r \]
Therefore,
\[ S(v,v,Tv,t)(1-r) \geq S(v,v,T^n v,v). \] (16)

**Case I:** \( 0 \leq r \leq \frac{1}{2} \) (hence \( \vartheta(r) = \frac{1-r}{r} \)) Hence,
\[ \max\{S(T^n v,v,T^n v,v), S(T^n v,T^n v,RT^n v,v)\}^{\vartheta(r)} \]
\[ \geq S(T^n v,v,T^n v,v)(\frac{1-r}{r})^r \]
\[ \geq S(T^n v,v,T^n v,v)^{1-r}. \]
So, by (16), we have:
\[ \max\{S(T^n v,v,T^n v,v), S(T^n v,T^n v,RT^n v,v)\}^{\vartheta(r)} \]
\[ \geq S(T^n v,v,T^n v,v)^{1-r} \]
\[ \geq S(v,v,Tv,t)^{1-r} \]
\[ \geq S(v,v,T^n v,v). \]
implies that
\[
\begin{align*}
\{ S(R^T v, R^T v, R v, t) & \ast S(T^{n+1} v, T^{n+1} v, T v, t) \ast \\
S(R^T v, R^T v, T v, t) & \ast S(R v, T^{n+1} v, T^{n+1} v, t) \} \\
& \geq S(T^n v, T^n v, T v, t)^r
\end{align*}
\]

Using (10), we obtain
\[
\begin{align*}
S(T v, T v, T^{n+1} v, t) & \geq S(v, v, T^n v, t)^r \\
& \geq S(v, v, T^n v, t)^r
\end{align*}
\]

So, the inequality (14) holds for each \( n \in \mathbb{N} \). Now \( T v \neq v \) and (14) implies that \( T^n v \neq v \) (If \( T^n v = v \), then we find \( S(T^n v, T^n v, v, t) \geq S(v, v, T v, t)^r \) implies:
\[
\begin{align*}
S(v, v, T v, t) & \leq S(v, v, T v, t)^r \\
& < S(v, v, T v, t). \text{ which is not possible.}
\end{align*}
\]

Hence, (13) implies that:
\[
\begin{align*}
S(v, v, T^{n+1} v, t) & \geq S(v, v, T^n v, t)^r \\
& \geq S(v, v, T^{n-1} v, t)^r
\end{align*}
\]

Hence, \( \lim_{n \to \infty} S(v, v, T^{n+1} v, t) = 1 \) this implies that \( T^n \to v \).
From this and (3.1.15), we have:
\[
\begin{align*}
S(T v, T v, v, t) & = \lim_{n \to \infty} S(T v, T v, T^n v, t) \\
& \geq \lim_{n \to \infty} S(v, v, T^n v, t)^r \\
& = S(T v, T v, v, t)^r.
\end{align*}
\]

Thus, \( S(T v, T v, v, t) = 1 \). Which is contrary to our assumption. Hence, \( T v = v \). As already proved \( v \) is fixed point of \( R \) also. Hence, \( T v = v = v \).

**Case II:** \( \frac{1}{\lambda^r} \leq r < 1 \) (hence \( \theta(r) = \frac{1}{1+r} \)). We will prove that there exist a subsequence \( \{ x_{n_k} \} \) of \( \{ x_n \} \) such that:
\[
\begin{align*}
S(x_{n_k}, x_{n_k}, T x_{n_k}, t)^{\frac{1}{1+r}} & \geq S(x_{n_k}, x_{n_k}, v, t), \\
or S(x_{n_k}, x_{n_k}, R x_{n_k}, t)^{\frac{1}{1+r}} & \geq S(x_{n_k}, x_{n_k}, v, t),
\end{align*}
\]

holds for each \( k \in \mathbb{N} \). Suppose that for every \( n \in \mathbb{N} \).

Then by (10) we have,
\[
\begin{align*}
\max\{ S(x_n, x_n, T x_n, t), S(x_n, x_n, R x_n, t) \}^{\frac{1}{1+r}} \\
& < S(x_n, x_n, v, t)
\end{align*}
\]

Hence,
\[
\begin{align*}
S(x_{n+1}, x_{n+1}, T x_{n+1}, t)^{\frac{1}{1+r}} & < S(x_{n+1}, x_{n+1}, v, t) \\
\text{and } S(x_{n+2}, x_{n+2}, R x_{n+2}, t) & < S(x_{n+2}, x_{n+2}, v, t)
\end{align*}
\]

holds for every \( n \in \mathbb{N} \).

Then by (10) we have:
\[
S(x_{2n+1}, x_{2n+1}, x_{2n+2}, t)^{\frac{1}{1+r}} \geq S(x_{2n+1}, x_{2n+1}, v, t) \ast S(x_{2n}, x_{2n}, v, t) \\
\geq S(x_{2n+1}, x_{2n+1}, T x_{2n+1}, t)^{\frac{1}{1+r}} \\
\ast S(x_{2n}, x_{2n}, T x_{2n}, t)^{\frac{1}{1+r}} \\
\geq S(x_n, x_n, x_{2n+1}, t)^{\frac{r}{1+r}} \ast S(x_{2n}, x_{2n}, x_{2n+1}, t)^{\frac{1}{1+r}} \\
= S(x_n, x_n, x_{2n+1}, t)
\]

which is impossible. Hence one of the following holds for each \( n \):
\[
S(x_{2n+1}, x_{2n+1}, T x_{2n+1}, t)^{\theta(r)} \geq S(x_{2n+1}, x_{2n+1}, v, t)
\]

Or
\[
S(x_{2n}, x_{2n}, R x_{2n}, t)^{\theta(r)} \geq S(x_{2n}, x_{2n}, v, t)
\]

If the following inequality hold:
\[
S(x_{2n+1}, x_{2n+1}, T x_{2n+1}, t)^{\theta(r)} \geq S(x_{2n+1}, x_{2n+1}, v, t)
\]

implies that
\[
\begin{align*}
\{ S(R x_{2n}, R x_{2n}, R v, t) & \ast S(T x_{2n}, T x_{2n}, T v, t) \ast \\
S(R x_{2n}, R x_{2n}, T v, t) & \ast S(R v, T x_{2n}, T x_{2n}, t) \} \\
& \geq S(x_{2n}, x_{2n}, v, t)^r
\end{align*}
\]

Hence,
\[
S(x_{2n+1}, x_{2n+1}, T v, t) = S(R x_{2n}, R x_{2n}, T v, t) \\
\geq S(x_{2n}, x_{2n}, v, t)^r
\]

Passing to the limit when \( n \to \infty \), we get that \( S(v, v, T v, t) \geq 1 \), which is possible only if \( T v = v \).
Similarly, if the following inequality hold.
\[
S(x_{2n}, x_{2n}, R x_{2n}, t)^{\theta(r)} \geq S(x_{2n}, x_{2n}, v, t)
\]

We have \( T v = v \). Thus, we have proved that \( v \) is a fixed point of \( T \).
Finally, we prove the uniqueness of the fixed point. Let \( v \) and \( u \) be common fixed points of \( T \) and \( R \), such that \( v \neq u \). Taking \( x, y = v \) and \( z = u \) in (2) and (3), we get:
\[
1 = \max\{ S(v, v, T v, t), S(v, v, R v, t) \}^{\theta(r)} \geq S(v, v, u, t)
\]

implies
\[
\begin{align*}
\{ S(R v, R v, R u, t) & \ast S(T v, T v, T u, t) \ast \\
S(R v, R v, T u, t) & \ast S(R u, T v, T u, t) \} \\
& \geq S(v, v, u, t)^r
\end{align*}
\]

Therefore, \( S(v, v, u, t) \geq S(v, u, u, t)^r \). which is not possible. Hence \( v = u \). This completes the proof.
Taking \( R = T \), we get the Suzuki type result.

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C. Corollary 3.2

Let \( (X, S, \ast) \) be a complete \( S \)-fuzzy metric space. If \( T : X \rightarrow X \) be a self mapping and \( \theta : [0, 1) \rightarrow \left( \frac{1}{2}, 1 \right) \) is defined by (1). Assume that there exists \( r \in [0,1) \) such that for each \( x, y, z \in X \),

\[
S(x, y, Tx, t)^{\theta(r)} \geq S(x, y, z, t)
\]

implies

\[
S(Tx, Ty, Tz, t) \geq S(x, y, z, t)^r
\]

Then \( T \) has a unique fixed point \( v \in X \).

D. Proof

To prove the corollary, It is enough to set \( R = T \) in Theorem (1).

E. Corollary 3.3

Let \( (X, S, \ast) \) be a complete \( S \)-fuzzy metric space. Let \( f, R, T : X \rightarrow X \) be three self-maps and \( \theta : [0, 1) \rightarrow \left( \frac{1}{2}, 1 \right) \) be defined by (2.1). Assume that there exists \( r \in [0,1) \) such that for each \( x, y, z \in X \),

\[
\max \{ S(x, y, fTx, t)S(x, y, fRx, t) \}^{\theta(r)} \geq S(x, y, z, t)
\]

implies

\[
\begin{align*}
&\{ S(Rx, fRy, fRz, t) * S(fTx, fTy, fTz, t) \ast S(fRx, fRy, fTz, t) \ast S(Ry, fRy, fTz, t) \} \\
&\geq S(x, y, z, t)^r
\end{align*}
\]

And if \( f \) is one-to-one, \( fR = Rf \) and \( fT = Tf \), then \( f, T \) and \( R \) have a unique common fixed point \( v \in X \).

F. Proof

Considering \( fR \) and \( fT \) as two maps in the given contractive condition of Theorem (3.1), we get a unique common fixed point for \( fR \) and \( fT \), i.e., \( fRv = fTv = v \).

Since \( f \) is one-to-one, \( fRv = fTv \) implies that \( Rv = Tv \).

From

\[
1 = \max \{ S(v, v, fTv, t), S(v, v, fRv, t) \}^{\theta(r)} \geq S(v, v, Tv, t)
\]

implies

\[
\begin{align*}
&\{ S(Rv, fRv, fRv, t) * S(fTv, fTv, fTv, t) \\
&\ast S(fRv, fRv, fTv, t) * S(RTv, fTv, fTv, t) \} \\
&= \{ S(fRv, fRv, fRv, t) * S(fTv, fTv, fTv, t) \\
&\ast S(fRv, fRv, fTv, t) * S(RTv, fTv, fTv, t) \} \\
&= \{ S(v, v, Rv, t) * S(v, v, Tv, t) \\
&\ast S(v, v, Tv, t) \} \\
&= S(v, v, v, t) \\
&\geq S(v, v, Tv, t)^r
\end{align*}
\]

Therefore \( Tv = v \), hence \( v \) is common fixed point of \( f, R \) and \( T \).

G. Corollary 3.4

Let \( (X, S, \ast) \) be a complete \( S \)-fuzzy metric space. Let \( T, R : X \rightarrow X \) be two self maps such that for each \( x, y, z \in X \) satisfying the following condition. If there exists \( r \in [0,1) \) such that for each \( x, y, z \in X \) satisfying the condition

\[
\max \{ S(x, y, fTx, t)S(x, y, fRx, t) \}^{\frac{1}{2}} \geq S(x, y, z, t)
\]

implies

\[
\begin{align*}
&\{ S(Rx, Ry, Rz, t) * S(Tx, Ty, Tz, t) \ast S(Rx, Rx, Ty, Tz, t) \ast S(Rx, Ry, Tz, t) \} \\
&\geq S(x, y, z, t)^r
\end{align*}
\]

Then there exists a unique common fixed point of \( T \) and \( R \).

H. Proof

It is enough to set \( r = \frac{1}{2} \) in Theorem 3.1, we get the result.

IV. CONCLUSION

In this article, we have proved some new type of Suzuki-type fixed point theorem for non-Archimedean \( S \)-fuzzy metric spaces which is generalization of Suzuki-Type fixed point results in \( S \)-fuzzy metric spaces. Also, we have shown the existence and uniqueness of fixed points for three self maps in the same structure.

REFERENCES

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