Edge Domination of the Nilpotent Cayley Graph of the Residue Class Ring \((\mathbb{Z}_n, \oplus, \odot)\)

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**ABSTRACT**

In their earlier study, the authors have obtained the properties of the nilpotent Cayley graph \(G(\mathbb{Z}_n, N)\) associated with the set of nilpotent elements \(N\) of the residue class ring \((\mathbb{Z}_n, \oplus, \odot)\) modulo an integer \(n \geq 1\) and its vertex domination parameters. In this paper, the edge cover and its associated parameters of this graph are found.

**Keywords:** Edge domination, Nilpotent Cayley graph, Vertex domination.

1. Introduction

During the mid-nineteenth century, the graph theoretic problem of finding a subset \(V_0\) of the vertex set \(V\) of a graph \(G\) with minimum cardinality, which covers every edge of \(G\), is similar to the problem of finding the minimum number of queens that can be placed on a chess board, so that all the squares are attacked by a queen or occupied by a queen. In the same way, one can think of finding a subset of edges of a graph \(G\) with minimum cardinality, which covers all vertices of \(G\) and these studies constitute an important branch of graph theory, namely, domination theory of graphs.

Berge \([1]\) and Ore \([2]\) gave a formal definition of vertex as well as edge dominating set, which are currently used in literature. Allan et al. \([3]\), \([4]\), Cockayne and Hedetniemi \([5]–[7]\), Haynes et al. \([8]\) and many others have contributed a lot to the domination theory of graphs.

Later Madhavi et al. \([9]–[11]\) studied domination parameters of graphs associated with some arithmetic functions.

The nilpotent graphs associated with a finite commutative ring \(R\) and the \(n \times n\) matrix ring \(M_n(R)\) are studied by Chen \([12]\), Nikmehr and Khojasteh \([13]\) and Basnet et al. \([14]\).

In \([15], [16]\) the authors have studied a new class of arithmetic Cayley graphs, namely, the nilpotent Cayley graphs associated with the set of nilpotent elements in the residue class ring \((\mathbb{Z}_n, \oplus, \odot), n \geq 1,\) an integer and its vertex domination.

Let \(N\) denote the set of nilpotent elements in the ring \((\mathbb{Z}_n, \oplus, \odot)\). The nilpotent Cayley graph of the ring \((\mathbb{Z}_n, \oplus, \odot)\) associated with the group \((\mathbb{Z}_n, \oplus)\) and its symmetric subset \(N\) is the graph \(G(\mathbb{Z}_n, N)\), whose vertex set \(V = \mathbb{Z}_n\) and the edge set:

\[E = \left\{(\bar{a}, \bar{b})/\bar{a}, \bar{b} \in \mathbb{Z}_n\text{ and either } \bar{a} - \bar{b} \in N, \text{ or, } \bar{b} - \bar{a} \in N\right\}.

In \([15]\), it is proved that, if \(n = \prod_{i=1}^{r} p_i^{\alpha_i}\), where \(p_i\)'s are primes such that \(p_1 < p_2 < \ldots < p_r\), \(\alpha_i \geq 1\) and \(1 \leq i \leq r\), are integers and \(m = p_1 p_2 p_3 \ldots p_r\), then the graph:
Since $0$ is a non-trivial nilpotent element, for $0 \in \mathbb{Z}_n$, $0.3$ is again a contradiction, since $0$ is a non-trivial nilpotent element.

The terminology and notations that are used in this paper can be found in [17] for graph theory [18], for algebra and [19] for number theory.

2. **Edge Cover of the Nilpotent Cayley Graph** $G(Z_n, N)$

A subset $E$ of the edges of a graph $G$, in which every vertex of $G$ is incident with some edge in $E$, is called an edge cover of the graph $G$. An edge cover of $G$, which contains the least number of edges is called a minimum edge cover of $G$. The number of edges in a minimum edge cover of $G$ is called the edge covering number of $G$ and it is denoted by $\beta'(G)$.

Let $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, where $p_1 < p_2 < \ldots < p_r$ are primes, $\alpha_i \geq 1$, $1 \leq i \leq r$ are integers. If $\alpha_r = 1$, for all $i$, then $m = p_1p_2 \ldots p_r$ and the ring $(\mathbb{Z}_n, \oplus, \odot)$ has no nilpotent elements, so that $N = \emptyset$. Hence the edge set $E$ of $G(Z_n, N)$ is also empty and one cannot think about the edge cover and the edge covering number of the graph $G(Z_n, N)$. So, throughout this paper, it is assumed that $\alpha_i > 1$, for atleast one $i, 1 \leq i \leq r$. Let $E_k = \left\{(\bar{m} + k, \bar{i} + m + \bar{k})/0 \leq i < j \leq (n - m)/m\right\}$.

In the following Theorems, we establish that $\bigcup_{k=0}^{m-n} E_k$ forms a minimum edge cover of the nilpotent Cayley graph $G(Z_n, N)$.

**Theorem 2.1:**
Each $E_k, 0 \leq k \leq m - 1$ contains distinct edges of the graph $G(Z_n, N)$.

**Proof:**
First let us show that each $E_k, 0 \leq k \leq m - 1$, contains distinct pairs of vertices of the graph $G(Z_n, N)$. Suppose that $(\bar{m} + k, \bar{i} + m + \bar{k}) = (\bar{m} + k, \bar{j} + m + \bar{k})$.

for some $\bar{m} + k, \bar{i} + m + \bar{k}, (\bar{m} + k, \bar{j} + m + \bar{k}) \in E_k, 0 \leq i < j \leq (n - m)/m$. Then, either

$$\bar{m} + k = \bar{m} + k$$

or,

$$\bar{m} + k = (\bar{i} + m + \bar{k})$$

From (1) we get, $\bar{m} + k = \bar{m} + k$, or, $\bar{i} = 0$. Since $0 \leq i < j \leq (n - m)/m < n$, we have $0 < j - i < n$. But $\bar{i} = 0$, for $0 < s < n$ and this leads to a contradiction.

On the other hand (2) gives:

$$\bar{m} + k = (\bar{i} + m + \bar{k}) = (\bar{i} + m + \bar{k} + m)$$

which on simplification gives $\bar{m} = 0$. This is again a contradiction, since $0 < 2m < n$. Thus each $E_k$ contains distinct order pairs of vertices of the graph $G(Z_n, N)$.

To see that each ordered pair in $E_k, 0 \leq k \leq m - 1$, represents an edge in the graph $G(Z_n, N)$, let: $(\bar{m} + k, \bar{i} + m + \bar{k}) \in E_k$, for $0 \leq i \leq n - m$.

Then:

$$\bar{m} + k = (\bar{i} + m + \bar{k}) = (\bar{i} + m + \bar{k}) - i(m + \bar{k}) - \bar{m} \in N,$$
so that \((jm + k, (j + 1)m + k)\) is an edge of the graph \(G(Z_n, N)\). Thus each \(E_k\) contains distinct edges of \(G(Z_n, N)\).

**Lemma 2.2:**
For \(0 \leq k < l \leq m - 1\), \(E_k \cap E_l = \emptyset\).

**Proof:**
If possible, assume that \(E_k \cap E_l \neq \emptyset\), for \(0 \leq k < l \leq m - 1\). Then, there exists an edge \((u, v) \in E_k \cap E_l\), or, \((u, v) \in E_k\) and \((u, v) \in E_l\). Then:
\[(u, v) = (im + k, (i + 1)m + k)\] and \((u, v) = (jm + l, (j + 1)m + l)\),
for some \(i, 0 \leq i, j \leq (n - m)/m\). For definiteness, we may assume that \(i < j\). This gives, either
\[im + k = jm + l\] and \((i + 1)m + k = (j + 1)m + l\),
(3)
or,
\[im + k = (j + 1)m + l\] and \((i + 1)m + k = jm + l\).
(4)
From (3), we get:
\[\frac{(j - i)m}{(i - 1)m} = \frac{l - k}{(l - k)}\]
(5)
But \((j - i)m\) is a nilpotent element of the ring \((Z_n, \oplus, \ominus)\) and \(0 \leq k < l \leq m - 1\) imply that \(l - k < m\). This implies that \((l - k)\) is not a nilpotent element of \((Z_n, \oplus, \ominus)\), since it contradicts the minimal property of \(m\). So, the (5) is not compatible. From (4), we get \(2m = 0\). But \(2m \neq 0\), since \(2m < n\), which again leads to a contradiction. So, our assumption that \(E_k \cap E_l \neq \emptyset\) is wrong and \(E_k \cap E_l = \emptyset\), for \(0 \leq k < l \leq m\).

**Theorem 2.3:**
The subset \(E_c = E_0 \cup E_1 \cup \ldots \cup E_{m-1}\), of the edge set \(G(Z_n, N)\) is an edge cover of the graph \(G(Z_n, N)\).

**Proof:**
By the Theorem 2.1, each \(E_k \subseteq E, 0 \leq k \leq m - 1\) and thus \(E_c = E_0 \cup E_1 \cup \ldots \cup E_{m-1} \subseteq E\), where
\[E_k = \left\{ (k, m + k), 2m + k, 3m + k), \ldots, (im + k, (i + 1)m + k), \ldots, (n - m + k, k) \right\}\]
Arranging the vertices occurring in the edge subsets \(E_1, E_2, E_3, \ldots, E_k, \ldots, E_{m-1}\) given above in the array of Fig. 2, one can easily observe that all the vertices of the graph \(G(Z_n, N)\) occur exactly once in this array, since \(E_k \cap E_l = \emptyset\), by the Lemma 2.2.

Let \(v\) be any vertex of the graph \(G(Z_n, N)\). A glance at the above array of the vertices of \(G(Z_n, N)\) shows that \(v = jm + k\) for \(j, 0 \leq j \leq (n - m)/m\) and \(k, 0 \leq k \leq m - 1\).
Thus, the vertex \(v\) is incident with the edge \((jm + k, (j + 1)m + k)\) of \(E_k \subseteq E_c\) and this shows that \(E_c\) is an edge cover of \(G(Z_n, N)\).
**Theorem 2.4:**

The subset $E_c = E_0 \cup E_1 \cup \ldots \cup E_m$ of the edge is a minimum edge cover of $G(Z_n, N)$.

**Proof:**

By the Theorem 2.3, $E_c = E_1 \cup E_2 \cup \ldots \cup E_\kappa \cup \ldots \cup E_m$, is an edge cover of the graph $G(Z_n, N)$. Consider the set $E_0 = E_c - \left\{ (m+k, (i+1)m+k) \right\}$, of the edge set of the graph $G(Z_n, N)$, got by deleting the edge $e = (m+k, (i+1)m+k)$ from $E_c$, for some $i$ and $k$, $0 \leq i \leq (n-m)/m$ and $0 \leq k \leq m-1$. The vertex $(m+k)$ as well as the vertex $(i+1)m+k$ of $G(Z_n, N)$ are not incident with any edge of $E_0$, as these vertices are incident with the edge $e = (m+k, (i+1)m+k)$ and this edge does not belong to $E_0$. This shows that $E_0$ is not an edge cover of the graph $G(Z_n, N)$, proving that $E_c$ is a minimum edge cover of the graph $G(Z_n, N)$.

**Theorem 2.5:**

The edge covering number $\beta'(G(Z_n, N))$ of $G(Z_n, N)$ is given by:

$$\beta'(G(Z_n, D_0)) = \begin{cases} \frac{n}{2}, & \text{if } 2^2/n \\ \frac{n+m}{2}, & \text{if } 2^2 \n. \end{cases}$$

**Proof:**

Let $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, where $p_1 < p_2 < \ldots < p_r$ are primes, $\alpha_i \geq 1, 1 \leq i \leq r$ are integers, $\alpha_i \geq 1$, $1 \leq i \leq r$, such that $\alpha_i > 1$, for at least one $i$ and let $m = p_1 p_2 \ldots p_r$. The following two cases arise.

**Case (i):**

Suppose $2^2 \mid n$. By the Theorem 2.4, the minimum edge cover $E_c$ of the graph $G(Z_n, N)$ is the disjoint union of the subsets $E_0, E_1, \ldots, E_k, \ldots, E_m$ of edges, where:

$$E_k = \{0(m+k), 1(m+k), 2(m+k), 3(m+k), \ldots, (n(m+k), (i+1)(m+k)) \}, \ldots, \left(\frac{n}{m} - 2\right) (m+k), \left(\frac{n}{m} - 1\right) (m+k)\}.$$

It is easy to see that,

$$\circ(E_0) = \circ(E_1) = \circ(E_2) = \ldots = \circ(E_{m-1}) = \frac{1}{2} \left[ \frac{n}{m} - 1 \right] + 1 = \frac{n + 2m}{2m},$$

and thus:

$$\circ(E_c) = \circ(E_0) + \circ(E_1) + \circ(E_2) + \ldots + \circ(E_{m-1}) = m \left( \frac{n + 2m}{2m} \right) = \frac{n}{2},$$

so that $\beta'(G(Z_n, N)) = \frac{n}{2}$.

**Case (ii):**

Let $2^2 \n n$. Again, by the Theorem 2.4, the minimum edge cover the graph $G(Z_n, N)$ is the disjoint union of the subsets $E_0, E_1, \ldots, E_k, \ldots, E_m$ of edges the graph $G(Z_n, N)$, where:

$$E_k = \{0(m+k), 1(m+k), 2(m+k), 3(m+k), \ldots, (i(m+k), (i+1)(m+k)) \}, \ldots, \left(\frac{n}{m} - 1\right) (m+k), \left(\frac{n}{m} - 1\right) (m+k)\}.$$

So,

$$\circ(E_0) = \circ(E_1) = \circ(E_2) = \ldots = \circ(E_{m-1}) = \frac{1}{2} \left( \frac{n}{m} - 1 \right) + 1 = \frac{n + m}{2m},$$

and thus:

$$\circ(E_c) = \circ(E_0) + \circ(E_1) + \circ(E_2) + \ldots + \circ(E_{m-1}) = m \left( \frac{n + m}{2m} \right) = \frac{n + m}{2}.$$

Hence edge covering number $\beta'(G(Z_n, N))$ of $G(Z_n, N)$ is $(n + m)/2$.

**Example 2.6:**

Consider the graph $G(Z_{18}, N)$, Here $n = 18 = 2 \cdot 3^2, m = 2 \cdot 3 = 6$ and $E_0 = \{0, 6, 12, 0\}$, $E_1 = \{1, 7, 13, 1\}$, $E_2 = \{2, 8, 14, 2\}$, $E_3 = \{3, 9, 15, 3\}$, $E_4 = \{4, 10, 16, 4\}$, and $E_5 = \{5, 11, 17, 5\}$, so that $E_c = \{0, 6, 12, 0, 1, 7, 13, 1, 2, 8, 14, 2, 3, 9, 15, 3, 4, 10, 16, 4, 5, 11, 17, 5\}$ is a minimum edge cover of $G(Z_{18}, N)$.
The edges in the above minimum edge cover of the graph $G(Z_{18}, N)$ are exhibited with boldface lines in the graph of $G(Z_{18}, N)$ given in Fig. 3. Observe that every vertex in the graph $G(Z_{18}, N)$ is incident with some edge in $E_c$.

Since $2^2 \nmid 18$, by the Theorem 2.5, the edge covering number $\beta'(G(Z_{n}, N))$ of the graph $G(Z_{18}, N)$ is $(18 + 6)/2 = 12$. This can be checked from the fact that the minimum edge cover $E_c$ of the graph $G(Z_{18}, N)$ contains 12 edges.

**Example 2.7:**

Consider the graph $G(Z_{27}, N)$. Here $n = 27 = 3^3$, $m = 3$ and

$$E_0 = \left\{ (0, 3), (6, 9), (12, 15), (18, 21), (24, 0) \right\},$$

$$E_1 = \left\{ (1, 4), (7, 10), (13, 16), (19, 22), (25, 1) \right\},$$

$$E_2 = \left\{ (2, 3), (8, 11), (14, 17), (20, 23), (26, 2) \right\},$$

so that

$$E_c = \left\{ (0, 3), (6, 9), (12, 15), (18, 21), (24, 0), (1, 4), (7, 10), (13, 16), (19, 22), (25, 1),
\right.$$  

$$\left( 2, 3), (8, 11), (14, 17), (20, 23), (26, 2) \right\}.$$

is a minimum edge cover of $G(Z_{27}, N)$ and the edges are in the above minimum edge cover are exhibited with boldface lines in the graph of $G(Z_{27}, N)$ given in Fig. 4. Since $2^2 \nmid 27$, by the Theorem 2.5, the edge covering number $\beta'(G(Z_{n}, N))$ of the graph $G(Z_{18}, N)$ is $(27 + 3)/2 = 15$. This can be checked from the fact that the minimum edge cover $E_c$ of the graph $G(Z_{27}, N)$ contains 15 edges.
Example 2.8:
Consider the graph $G(Z_{36}, N)$. Here $n = 36 = 2^2 \cdot 3^2$ and $m = 6$. So

$$E_0 = \left\{ \left( \overline{0}, \overline{6} \right), \left( 12, 18 \right), \left( 24, 30 \right) \right\}, \quad E_1 = \left\{ \left( \overline{1}, \overline{7} \right), \left( 13, 19 \right), \left( 25, 31 \right) \right\},$$

$$E_2 = \left\{ \left( 2, 8 \right), \left( 14, 20 \right), \left( 26, 32 \right) \right\}, \quad E_3 = \left\{ \left( 3, 9 \right), \left( 15, 21 \right), \left( 27, 33 \right) \right\},$$

$$E_4 = \left\{ \left( 4, 10 \right), \left( 16, 22 \right), \left( 28, 34 \right) \right\} \quad \text{and} \quad E_5 = \left\{ \left( 5, 11 \right), \left( 17, 23 \right), \left( 29, 35 \right) \right\},$$

so that

$$E_2 = \left\{ \left( \overline{0}, \overline{6} \right), \left( 12, 18 \right), \left( 24, 30 \right), \left( 1, 7 \right), \left( 13, 19 \right), \left( 25, 31 \right), \left( 2, 8 \right), \left( 14, 20 \right), \left( 26, 32 \right), \left( 3, 9 \right), \left( 15, 21 \right), \left( 27, 33 \right), \left( 4, 10 \right), \left( 16, 22 \right), \left( 28, 34 \right), \left( 5, 11 \right), \left( 17, 23 \right), \left( 29, 35 \right) \right\}$$

is a minimum edge cover of $G(Z_{36}, N)$ and the edges in the above minimum edge cover of the graph $G(Z_{36}, N)$ are exhibited with boldface lines in the graph of $G(Z_{36}, N)$ given in Fig. 5. Since $2^2/36$, by the Theorem 2.5, the edge covering number $\beta'(G(Z_n, N))$ of the graph $G(Z_{36}, N)$ is $36/2 = 18$. This can be checked from the fact the minimum edge cover $E_2$ of the graph $G(Z_{36}, N)$ contains 18 edges.

3. **Edge Domination of the Nilpotent Cayley Graph $G(Z_n, N)$**

Let $G$ be a graph. A subset $F$ of the edge set $E$ of $G$ is called an edge dominating set of $G$, if each edge in $E - F$ is adjacent to some edge in $F$. An edge domination set of a graph $G$, consisting of minimum number edges of $G$ is called a minimum edge dominating set of $G$. The number of edges in a minimum edge dominating set is called the edge domination number of $G$ and it is denoted by $\gamma'(G)$. An edge dominating set and the related parameters of the graph $G(Z_n, N)$ are determined according as $2^2/n$ and $2^2 \cdot 1/n$.

**Theorem 3.1:**
If $2^2/n$, then the subset

$$E_d = \left\{ \left( \overline{0}, \left( \frac{n}{2} \right) \right), \left( \overline{1}, \left( \frac{n}{2} + 1 \right) \right), \left( \overline{2}, \left( \frac{n}{2} + 2 \right) \right), \ldots, \left( \overline{i}, \left( \frac{n}{2} + i \right) \right), \ldots, \left( \overline{\frac{n}{2} - 1}, \frac{n}{2} - 1 \right) \right\},$$

is of ordered pairs of vertices of the graph $G(Z_n, N)$ contains distinct edges of the graph $G(Z_n, N)$.

**Proof:**
Let $2^2/n$ and let

$$E_d = \left\{ \left( \overline{0}, \left( \frac{n}{2} \right) \right), \left( \overline{1}, \left( \frac{n}{2} + 1 \right) \right), \left( \overline{2}, \left( \frac{n}{2} + 2 \right) \right), \ldots, \left( \overline{i}, \left( \frac{n}{2} + i \right) \right), \ldots, \left( \overline{\frac{n}{2} - 1}, \frac{n}{2} - 1 \right) \right\}.$$

For $(\overline{i}, (\frac{n}{2} + i)), (\overline{j}, (\frac{n}{2} + j)) \in E_d$, let $(\overline{i}, (\frac{n}{2} + i)) = (\overline{j}, (\frac{n}{2} + j))$, for some $i, j, 0 \leq i < j \leq n/2 - 1$. This gives $i = j, j - i = 0$. Since $0 \leq i < j \leq n/2 - 1 < n$, this is a contradiction to the fact that $\overline{s} \neq \overline{0}$ for any positive integer $s < n$. So $E_d$ contains distinct ordered pairs. Also, for $(\overline{i}, (\frac{n}{2} + i)) \in E_d$, for $0 \leq i \leq n/2 - 1$, we have $(\frac{n}{2} + i - i = \frac{n}{2})$.
Since $2^2/n$, we have $n = 2^ia_1p_1^{a_1}p_2^{a_2} \ldots p_r^{a_r}$, where $2 < p_2 < \ldots < p_r$ are primes, $a_1 \geq 2$ and $a_i \geq 1$, for $2 \leq i \leq r$, are integers. So $n/2 = 2^{a_1-1}p_1^{a_1}p_2^{a_2} \ldots p_r^{a_r}$ and $a_1 - 1 \geq 1$ and $a_i \geq 1$, for $2 \leq i \leq r$, so that $n/2$ is an integer. But $(n/2)$ is a nilpotent element of the ring $(\mathbb{Z}_n, \oplus, \odot)$, so that $(l, n/2 + i)$ is an edge of $G(Z_n, N)$. Thus, $E_d$ contains distinct edges of the graph $G(Z_n, N)$. 

**Theorem 3.2:**

If $2^2|n$, then the subset

$$E_d = \left\{ \left( \overline{0}, \left( \frac{n}{2} \right) \right), \left( \overline{1}, \left( \frac{n}{2} + 1 \right) \right), \left( 2, \left( \frac{n}{2} + 2 \right) \right), \ldots, \left( i, \left( \frac{n}{2} + i \right) \right), \ldots, \left( \frac{n}{2} - 1, n - 1 \right) \right\}$$

of the edge set of $G(Z_n, N)$ is a minimum edge dominating set of the graph $G(Z_n, N)$.

**Proof:**

By Theorem 3.1, $E_d$ is a subset of edges the graph $G(Z_n, N)$. By arranging the vertices occurring in the edges of $E_d$ in the following way, one can observe that all the vertices of the graph $G(Z_n, N)$ are covered exactly once by the edges in $E_d$.

Let $(\overline{a}, \overline{b})$ be an edge in $E - E_d$. Since the vertices of $G(Z_n, N)$ are exactly the vertices occurring in the edges $E_d$ (as in Fig. 6), it follows that either $\overline{a} = \overline{i}$ for some $i$, $0 \leq i \leq n/2 - 1$, or, $\overline{b} = \frac{n}{2} + j$, for some $j$, $0 \leq j \leq n/2 - 1$. In the first case the edge $(\overline{a}, \overline{b})$ in $E_d$ is incident with the edge $(\overline{n/2 + i}, \overline{i})$ in $E_d$ and in the second case, the edge $(\overline{a}, \overline{b})$ in $E_d$ is incident with the edge $(\overline{n/2 + j}, \overline{j})$ in $E_d$. So, $E_d$ is an edge dominating set of $G(Z_n, N)$.

Let $E_1$ be the edge set got by deleting one edge $(\overline{i}, \overline{n/2 + i})$ from the edge dominating set $E_d$, that is $E_1 = E_d - \left\{ \left( \overline{i}, \overline{n/2 + i} \right) \right\}$. Clearly, the edge $(\overline{i}, \overline{n/2 + i}) \notin E_1$, so that $(\overline{i}, \overline{n/2 + i}) \in E - E_1$. Further this edge $(\overline{i}, \overline{n/2 + i})$ is not adjacent to any edge of $E_1$, so that $E_1$ is not an edge dominating set of the graph $G(Z_n, N)$. These show that $E_d$ is minimum edge dominating set of $G(Z_n, N)$.

**Theorem 3.3:**

If $2^2 \nmid n$, then the subset

$$E_d = \left\{ \left( \overline{0}, \left( \frac{n-m}{2} \right) \right), \left( \overline{1}, \left( \frac{n-m}{2} + 1 \right) \right), \ldots, \left( i, \left( \frac{n-m}{2} + i \right) \right), \ldots, \left( \frac{n-m}{2} - 1, \left( \frac{n-m}{2} + \frac{n-m}{2} - 1 \right) \right) \right\}$$

of ordered pairs of vertices of the graph $G(Z_n, N)$ contains distinct edges of the graph $G(Z_n, N)$.

**Proof:**

Let $2^2 \nmid n$. As in the Theorem 3.1, one can see that, $E_d$ contains distinct ordered pairs of vertices of $G(Z_n, N)$. To see that each ordered pair in $E_d$ represents an edge of the graph $G(Z_n, N)$. Let $\left( \overline{i}, \left( (n-m)/2 + i \right) \right) \in E_d$, for $0 \leq i \leq n/2 - 1$. Then $(n-m)/2 + i = \left( (n-m)/2 \right)$. Since $2^2 \nmid n$, either $n$ is even (if $2|n$), or, odd (if $2 \nmid n$). Suppose that $n$ is even say, then $n = 2p_2^{\alpha_2}p_3^{\alpha_3} \ldots p_r^{\alpha_r}$, where $2 < p_2 < \ldots < p_r$ are primes, $\alpha_i \geq 1$, for $2 \leq i \leq r$, are integers, and at least one $\alpha_i > 1$, for $2 \leq i \leq r$. Now:

$$n - m = m \prod_{i=2}^{r} p_i^{\alpha_i-1} - m = m \left[ \prod_{i=2}^{r} p_i^{\alpha_i-1} - 1 \right].$$

Since $p_2, p_3, \ldots, p_r$ are all odd, $\prod_{i=2}^{r} p_i^{\alpha_i-1}$ is odd and hence $\prod_{i=2}^{r} p_i^{\alpha_i-1} - 1$ is even. So, let $\prod_{i=2}^{r} p_i^{\alpha_i} - 1 = 2K$, for some positive integer $K$. Then $(n-m)/2 = (m \cdot 2K)/2 = mK$. 

Vol 4 | Issue 5 | October 2023
Since $\overline{m}$ is a nilpotent element of the ring $(\mathbb{Z}_n, ⋆, \circ)$, it follows that $K\overline{m}$, or $(\overline{(n-m)/2})$ is also a nilpotent element of the ring $(\mathbb{Z}_n, ⋆, \circ)$. This shows that $(\overline{i}, (n-m)/2 + \overline{i})$, $0 \leq i \leq n/2 - 1$ is an edge of the graph $G(\mathbb{Z}_n, N)$.

Let $n$ be odd and let $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, where $p_1 < p_2 < \ldots < p_r$ are all odd primes, $\alpha_i \geq 1$, $1 \leq i \leq r$ are integers, such that $\alpha_i > 1$, for at least one $i$. Then all $p_i$’s are odd and

$$n - m = \prod_{i=1}^{r} p_i^{\alpha_i} - m = m \left[ \prod_{i=1}^{r} p_i^{\alpha_i - 1} - 1 \right].$$

Since $p_1, p_2, \ldots, p_r$ are all odd, $\prod_{i=1}^{r} p_i^{\alpha_i - 1}$ is odd and hence $\prod_{i=1}^{r} p_i^{\alpha_i - 1} - 1$ is even. So $\prod_{i=1}^{r} p_i^{\alpha_i - 1}$, $-1 = 2M$ for some positive integer $M$. Then $(n-m)/2 = (m \cdot 2M)/2 = mM$.

Since $\overline{m} \in N$, implies that $(n-m)/2 = M\overline{m} \in N$, and thus $(\overline{i}, (n-m)/2 + \overline{i}), 0 \leq i \leq n/2 - 1$, is an edge of the graph $G(\mathbb{Z}_n, N)$.

**Theorem 3.4:**

If $2^2 \nmid n$, then the subset

$$E_d = \left\{ \left(\overline{0}, \left(\frac{n-m}{2}\right)\right), \left(\overline{1}, \left(\frac{n-m}{2} + 1\right)\right), \ldots, \left(\overline{i}, \left(\frac{n-m}{2} + i\right)\right), \ldots, \left(\left(\frac{n-m}{2} - 1\right), \left(\frac{n-m}{2} + \frac{n-m}{2} - 1\right)\right) \right\},$$

of the edge set of the graph $G(\mathbb{Z}_n, N)$ is a minimum edge dominating set of the graph $G(\mathbb{Z}_n, N)$.

**Proof:**

Consider the subset

$$E_d = \left\{ \left(\overline{0}, \left(\frac{n-m}{2}\right)\right), \left(\overline{1}, \left(\frac{n-m}{2} + 1\right)\right), \ldots, \left(\overline{i}, \left(\frac{n-m}{2} + i\right)\right), \ldots, \left(\left(\frac{n-m}{2} - 1\right), \left(\frac{n-m}{2} + \frac{n-m}{2} - 1\right)\right) \right\},$$

of the edge set of the graph $G(\mathbb{Z}_n, N)$. By arranging the vertices occurring in the edges of $E_d$ in the following way, one can observe that all the vertices of the graph $G(\mathbb{Z}_n, N)$ are covered by the edges in $E_d$.

Let $(\overline{a}, \overline{b})$ be an edge in $E - E_d$. Since the vertices of $G(\mathbb{Z}_n, N)$ are exactly the vertices occurring in the edges in $E_d$ (see Fig. 7), it follows that, either $\overline{a} = \overline{i}$, for some $i, 0 \leq i \leq (n-m)/2 - 1$, or, $\overline{b} = (n-m)/2 + \overline{j}$, for some $j, 0 \leq j \leq (n-m)/2 - 1$. In the first case the edge $(\overline{a}, \overline{b})$ in $E_d$ is incident with the edge $(n-m)/2 + \overline{i}, \overline{j})$ in $E_d$. In the second case the edge $(\overline{a}, \overline{b})$ in $E_d$ is incident with the edge $(\overline{n-m)/2 + j}, \overline{i})$ in $E_d$. So, $E_d$ is an edge dominating set of the graph $G(\mathbb{Z}_n, N)$. Let $E_1$ be the subset of edges set by deleting the edge $(\overline{i}, (n-m)/2 + \overline{i}) \in E_d$, that is $E_1 = E_d - \left\{ \left(\overline{i}, (n-m)/2 + \overline{i}\right) \right\}$.

Then, the edge $(\overline{i}, (n-m)/2 + \overline{i}) \notin E_1$, so that $(\overline{i}, (n-m)/2 + \overline{i}) \in E - E_1$. Further this edge $(\overline{i}, (n-m)/2 + \overline{i})$ is not adjacent to any edge of $E_1$. So $E_1$ is not an edge dominating set of the graph $G(\mathbb{Z}_n, N)$ and $E_d$ is a minimum edge dominating set of the graph $G(\mathbb{Z}_n, N)$.

**Theorem 3.5:**

The edge domination number $\gamma'(G(\mathbb{Z}_n, N))$ of $G(\mathbb{Z}_n, N)$ is given by:
Edge Domination of the Nilpotent Cayley Graph of the Residue Class Ring \((\mathbb{Z}_n, \oplus, \odot)\)

Madhavi et al.

Proof:

Example 3.7:

Let \(n = \prod_{i=1}^{r} p_i^{a_i}\), where \(p_1 < p_2 < \ldots < p_r\) are primes \(a_i \geq 1\) and \(1 \leq i \leq r\) are integers such that \(a_i \geq 1\) for at least one \(i\), and let \(m = p_1 p_2 \ldots p_r\).

Let \(2^2/n\). By the Theorem 3.2, the minimum edge dominating set \(E_d\) of \((\mathbb{Z}_n, N)\) is given by:

\[
E_d = \left\{ \left[0, \left(\frac{n}{2}\right)\right], \left[1, \left(\frac{n}{2} + 1\right)\right], \ldots, \left[i, \left(\frac{n}{2} + i\right)\right], \ldots, \left[\frac{n}{2} - 1, \frac{n - 1}{2}\right] \right\}.
\]

So, \(\alpha(E_d) = (n/2 - 1) + 1 = n/2\) and the minimum edge domination number \(\gamma'(G(\mathbb{Z}_n, N))\) is \(n/2\).

Let \(2^2 \nmid n\). Then by the Theorem 3.4, the minimum edge dominating set \(E_d\) of \((\mathbb{Z}_n, N)\) is given by

\[
E_d = \left\{ \left[0, \left(\frac{n - m}{2}\right)\right], \left[1, \left(\frac{n - m}{2} + 1\right)\right], \ldots, \left[i, \left(\frac{n - m}{2} + i\right)\right], \ldots, \left[\frac{n - m}{2} - 1, \frac{n - m}{2} - 1\right] \right\}.
\]

So, \(\alpha(E_d) = (n - m)/2 - 1 + 1 = (n - m)/2\), and thus the minimum edge domination number \(\gamma'(G(\mathbb{Z}_n, N))\) is \((n - m)/2\).

Example 3.6:

Let us consider the graph \((\mathbb{Z}_{36}, N)\). Here \(36 = 2^2 \cdot 3^2\), so that \(2^2/n\) and \(m = 6\). By the Theorem 3.2, the minimum edge dominating set \(E_d\) of the graph \((\mathbb{Z}_{36}, N)\) is given by \(E_d = \left\{ [0, 18], [1, 19], [2, 20], [3, 21], [4, 22], [5, 23], [6, 24], [7, 25], [8, 26], [9, 27], \ldots, [10, 30], [11, 31], [12, 32], [13, 33], [14, 34], [15, 35] \right\} \).

Since \(E_d\) contains 18 edges, the edge domination number of \((\mathbb{Z}_{36}, N)\) is \(36/2\).

The edges in the minimum edge dominating set \(E_d\) of the graph \((\mathbb{Z}_{36}, N)\) are exhibited by boldface lines in the graph of \((\mathbb{Z}_{36}, N)\) given in Fig. 8.

Example 3.7:

Let us consider the graph \((\mathbb{Z}_{18}, N)\). Here \(18 = 2 \cdot 3^2\), so that \(2^2 \nmid 18\) and \(n\) is even. Further \(m = 6\). By the Theorem 3.4, the minimum edge dominating set \(E_d\) of the graph \((\mathbb{Z}_{18}, N)\) is given by \(E_d = \left\{ [0, 6], [1, 7], [2, 8], [3, 9], [4, 10], [5, 11] \right\} \). Since \(E_d\) contains 6 edges, the edge domination number of \((\mathbb{Z}_{18}, N)\) is \(6/2\).

The edges in the minimum edge dominating set \(E_d\) of the graph \((\mathbb{Z}_{18}, N)\) are exhibited by boldface lines in the graph of \((\mathbb{Z}_{18}, N)\) given in Fig. 9.
Example 3.8:

Let us consider the graph $G(Z_{45}, N)$. Here $45 = 3^2 \cdot 5$, so that $2^2 \nmid 45$ and $n$ is odd. Further $m = 15$. By the Theorem 3.4, the minimum edge dominating set $E_d$ of the graph $G(Z_{45}, N)$ is given by $E_d = \{ (0, 15), (1, 16), (2, 17), (3, 18), (4, 20), (5, 22), (7, 23), (8, 24), (10, 25), (11, 26), (12, 27), (13, 28), (14, 29) \}$. Since $E_d$ contains 15 edges, the edge domination number of $G(Z_{45}, N)$ is $15 = (45 - 15)/2$.

The edges in the minimum edge dominating set $E_d$ of the graph $G(Z_{45}, N)$ are exhibited by boldface lines in the graph of $G(Z_{45}, N)$ given in Fig. 10.

Example 3.9:

Let us consider the graph $G(Z_{27}, N)$. Here $27 = 3^3$, which is a power of a single prime 3. So that $2^2 \nmid 27$ and 27 is odd. Further $m = 3$. By the Theorem 3.4, the minimum edge dominating set $E_d$ of the graph $G(Z_{27}, N)$ is given by $E_d = \{ (0, 12), (1, 13), (2, 14), (3, 15), (4, 16), (5, 17), (6, 18), (7, 19), (8, 20), (9, 21), (10, 22), (11, 23) \}$. Since $E_d$ contains 12 edges, the edge domination number of $G(Z_{27}, N)$ is $12 = (27 - 3)/2$.

The edges in the minimum edge dominating set $E_d$ of the graph $G(Z_{27}, N)$ are exhibited by boldface lines in the graph of $G(Z_{27}, N)$ given in Fig. 11.
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Conflict of Interest

Authors declare that they do not have any conflict of interest.

References