Higher Order Moments of the Order Statistics for the Rectangular, Exponential, Gamma and Weibull Distributions

Sampath Kumar and V. V. Hara Gopal

Abstract — In this paper we discuss the problem of Higher Order Moments for the order Statistics for the Rectangular, Exponential, Gamma and Weibull distributions by finding the order statistic distributions for the base distribution and modified distributions, the base distribution is to deduce the corresponding distribution by the polynomial modifier. These higher order moments are very much useful in most of the Data sciences and Image analysis.

Key words — Higher order Moments, Base and Modifier Distributions.

I. INTRODUCTION

In this paper we consider the procedure of finding out the higher order moments [4]-[6] for order statistic distribution for Rectangular, Exponential, Gamma and Weibull distributions. For these distributions we first find out the order statistic distribution and treat this as the base distribution and modify the corresponding distribution by considering the polynomial modifier [7], [13] that is,

$$f_m(x) = (n.f(x)(F(x))^{n-1}) \{1 + \alpha g(x_n)\} \quad (1)$$

This gives us the modifier expression of the maximum order statistic distribution. Similarly, we consider the modifier of the minimum order statistic distribution as:

$$f_m(x) = (n.f(x)(1-F(x))^{n-1}) \{1 + \alpha g(x_n)\} \quad (2)$$

With the above two modifier functions we compute the higher order moments for base and as well as for the modifier distributions.

For these Minimum and Maximum order statistic distributions, higher order moments [8] are computed by considering different sample sizes, that is $n=2, 3, 5$ and 10 these are computed by assuming that the first four moments of these order statistic distributions are made equal and the higher order moments computed, which are different from base. As the sample size increases the moment differences are increasing to some extent, this is true for all the sample sizes considered. This aspect is discussed in detail in the following sections.

II. DISTRIBUTIONS WHOSE FIRST K MOMENTS ARE IDENTICAL

Let $f_i(x)$ and $f_m(x)$ be two probability density functions of random variates whose first k moments are the same, but whose higher order moments differ. Then, we can write

$f_m(x) = f_i(x) \{1 + \alpha g(x_n)\}$, where $g(x_n)$ is a “modifier” function such that $f_i(x) g(x_n)$ is bounded over the domain of definition of x ,

Integrates to zero and, α is a suitably chosen modifier weight.

Since the first k moments should be the same for both the distributions, one should have:

$$\int_x f_m(x) x^r dx = \int_x f_i(x) x^r dx \text{ for } r = 1, 2, 3, \dots, k$$

i.e., one should have:

$$\int_x f_i(x) g(x_n) x^r dx = 0 \text{ for } r = 1, 2, 3, \dots, k$$

This fact enables one to generate a class of functions $f_m(x)$, with $f_i(x)$ (included in this class) as its base function; all these functions are to have their first k moments the same as the corresponding moments of the base distribution.

It is not difficult to envisage such a function $g(x_n)$. An obvious choice is supplied by orthogonal system of functions, with $f_i(x)$ as the kernel. In particular, $g(x_n)$ may be a polynomial $P_m(x)$ of degree more than k if x has a finite range and $P_m(x).w(x)$, where $w(x)$ is a suitable dampening function like e^{-x} or e^{-x^2} if the range of x is not finite. That is if x has a finite range, we will take $w(x)=1$ and if x has an infinite range we will take $w(x)=e^{-x}$ or e^{-x^2} . We take here that ‘ x ’ has a finite range for Rectangular variate and an infinite range ($0 \leq x < \infty$) for Exponential variate, so here we take for this variate, $g(x_n)=P_m(x).w(x)$.

Let $P_m(x)$ be a polynomial defined as $b_0 + b_1x + b_2x^2 + b_3x^3 + \dots + b_mx^m$ such that:

$$\int_x f_i(x) P_m(x) w(x) x^r = 0 \text{ for } r = 0, 1, 2, \dots, k$$

It is obviously possible to obtain such polynomial $P_m(x)$ for $f_i(x)$, if the first k moments exist. Also, under the same assumptions, it is obvious that $g(x_n) f_i(x)$ is finite for all valid values of x .

Let $\theta = \max_x |P_m(x)w(x)|$ be the maximum absolute value that $P_m(x)w(x)$ takes over the range of interest.

Then $f_m(x) = f_i(x) \{1 + \alpha(g(x_n) = P_m(x)w(x))\}$ will be non negative for all x values if $|\alpha| < \frac{1}{\theta}$ and we will have $f_m(x)$ as

a class of density functions which have the first ‘ k ’ moments as $f_i(x)$ does.

The difference between the r^{th} moment of $f_m(x)$ and $f_i(x)$, is defined as Δ_r .

$$\begin{aligned}\Delta_r &= \int_x \{(f_m(x) - f_i(x))\} x^r dx \\ &= \alpha \left\{ \int_x f_i(x) P_m(x) w(x) x^r dx \right\} \\ &= \alpha \left\{ \sum_{s=0}^m b_s \int_x f_i(x) w(x) x^{r+s} dx \right\}\end{aligned}$$

i.e. $\Delta_r = \alpha \left\{ \sum_{s=0}^m b_s \mu'_{s+r} \right\}$ where μ'_{s+r} is the moment of order $(r+s)$, of the density function $f_i(x) w(x)$.

In other words, one has:

$$\alpha (b_0 \mu'_r + b_1 \mu'_{r+1} + b_2 \mu'_{r+2} + b_3 \mu'_{r+3} + \dots + b_m \mu'_{r+m}) = \Delta_r, r = 0, 1, \dots$$

Using matrix notation, one can restate this result as follows:

$$M = \begin{bmatrix} \mu'_0 & \mu'_1 & \dots & \mu'_m \\ \mu'_1 & \mu'_2 & \dots & \mu'_{m+1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \mu'_k & \mu'_{k+1} & \dots & \mu'_{m+k} \\ \mu'_{k+1} & \mu'_{k+2} & \dots & \mu'_{m+k+1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \bar{b} = \begin{bmatrix} b_0 \\ b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix}$$

$$\text{and } \bar{\Delta} = \begin{bmatrix} \Delta_0 \\ \Delta_1 \\ \cdot \\ \cdot \\ \cdot \\ \Delta_k \\ \Delta_{k+1} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

Since the moment differences $\Delta_0, \Delta_1, \dots, \Delta_k$ are to be equal to zero. One should have

$$\alpha .M. \bar{b} = \bar{o}$$

where M is of order $(k+1, m+1)$, the coefficient vector $\bar{b}$ of order $m+1$ and $\bar{o}$ is of order $k+1$.

Since the equation system is homogeneous, one can take $b_m = 1$. The rank of the matrix M is $1 + \min(k, m)$. Hence, if $k=m$, the only solution is the trivial solution $\bar{b} = \bar{o}$. Hence,

m , the degree of the polynomial, is to be greater than k .

Taking $k=4$, that is the first four moments should be the same, one should have $m \geq 5$.

The solution set $\bar{b}$ is a vector space of dimension $(m-k)$, and any linear combination of these solutions will give an acceptable polynomial $P_m(x)$. As already noted, (since the polynomial is bounded in the range of interest) proper choice of the modifier-scalar α values makes $1 + \alpha P_m(x)w(x)$ non-negative in the region of interest, i.e. $f_m(x) = f_i(x) \{1 + \alpha P_m(x)w(x)\}$ will be density function, all having the first k moments same with $f_i(x)$. The other higher moments can easily be obtained as functions of the 'b'-coefficients as chosen above.

We shall now construct a family of distributions having the first four moments same as those of the base distribution $f_n(x)$, where $i=1, 2, 3, \dots, n$ defined by:

$$f_1(x) = \frac{d(F(x))}{dx}, f_2(x) = \frac{d((F(x))^2)}{dx}, \dots,$$

$$f_n(x) = \frac{d((F(x))^n)}{dx}$$

$$f_n(x) = \frac{d((F(x))^n)}{dx} = \frac{d}{dx}(F_n(x)) = n.f(x)(F(x))^{n-1}$$

where $F_2(x) = (F(x))^2$ that is the square of the distribution of 'x', where the squaring process is monotonic and

$f_2(x) = \frac{d}{dx}(F_2(x)) = 2.f(x)(F(x))$ is the density of 'x' the "second iterate" that is the 2nd order statistic distribution is considered and the standardized moments are computed. In the next section we consider various distributions of order statistic and their moments.

III. RECTANGULAR Distribution

We shall now consider the Rectangular Distribution, defined by:

$$F(x) = x, 0 \leq x \leq 1,$$

and now the Minimum Order statistic density function of Rectangular Distribution is:

$$f_n(x) = n.(1-x)^{n-1};$$

and Maximum Order statistic density function of Rectangular Distribution is:

$$f_n(x) = n.x^{n-1}$$

where 'n' is the sample size.

Now we consider the polynomials of order 5 as the modifier i.e. $k=4, m=5$ (polynomial order), one gets M , such that:

$$\alpha .M. \bar{b} = \bar{\Delta}$$

Solving the above equation for ' $\bar{b}$ ', we get the modifier polynomial coefficients as ' $\bar{b}$ ', from which we get the

following 5th degree polynomial in 'x' with 'n', the sample size.

A. Minimum Order Statistic for Rectangular Distribution

The following Table I gives the polynomials and the corresponding 'θ' values for a chosen value of the 'n'.

TABLE I: POLYNOMIALS AND THE CORRESPONDING 'θ' VALUES FOR A CHOSEN VALUE OF THE 'N'

n	$g(x_n)$	$\theta = \max_x g(x_n) $
2	$-0.002165+0.07575876*x-$ $0.606061*x^2+1.818182*x^3-$ $2.272727*x^4+x^5$	0.012987
3	$-0.001263+0.050505*x-$ $0.454546*x^2+1.515152*x^3-$ $2.083333*x^4+x^5$	0.026515
5	$-0.0004995+0.024975*x-$ $0.274725*x^2+1.098901*x^3-$ $1.785714*x^4+x^5$	0.062937
10	$-0.000086+0.0064499*x-$ $0.103199*x^2+0.584795*x^3-$ $1.315789*x^4+x^5$	0.172171

Table II below gives the 'α' for the different 'n' and 'θ' values.

TABLE II: α VALUES FOR THE DIFFERENT 'N' AND 'θ' VALUES

n	2	3	5	10
θ	0.012987	0.026515	0.062937	0.172171
$\alpha = \frac{1}{\theta}$	77.00	37.714	15.88889	5.808192

Thus, the general formula for modified minimum order statistic distributions is:

$$f_m(x) = (n.f(x)(1-F(x))^{n-1}) \{1 + \alpha g(x_n)\}$$

Thus, for instance, when n =2, we have:

$$f_m(x) = 2.(1-x).\{1 + \alpha g(x_n)\}, \text{ and } \alpha \leq 77.00$$

When n =3, we have $f_m(x) = 3.(1-x)^2.\{1 + \alpha g(x_n)\}$, and $\alpha \leq 37.714$. When n =5, we have $f_m(x) = 5*(1-x)^4.\{1 + \alpha g(x_n)\}$, and $\alpha \leq 15.889$ and when n =10, we have $f_m(x) = 10*(1-x)^9.\{1 + \alpha g(x_n)\}$, and $\alpha \leq 5.809$.

Similarly, $f_m(x)$ can be computed for any value of 'n'.

Table III (a), Table III (b), Table III (c) and Table III (d) clearly show that the first four standardized moments (first two moments are actual moments, i.e., mean and variance) are same for base and modified distributions and higher order moments are different.

Standardized Moments for base, modified distribution and their difference.

TABLE III (A): FOR THE CASE OF N=2 AND α = 77.00

Moment	Base	Modified	Difference
2	0.0556	0.0556	0
3	0.565685	0.565685	0
4	2.4	2.4	0
5	3.232488	3.315138	0.082650
6	8.857143	9.126876	0.269733
7	16.970563	18.03865	1.068087
8	41.6	44.8224	3.2224
9	92.155298	102.08848	9.933182
10	222.54546	251.06721	28.52175
11	525.6523	607.20358	81.55128
12	1283.517	1511.2094	227.6924
13	3134.22	3766.2267	632.0067
14	7772.8	9510.5278	1737.7278
15	19377.1	24135.983	4758.883

TABLE III (B): FOR THE CASE OF N=3 AND α = 37.714

Moment	Base	Modified	Difference
1	0.25	0.25	0
2	0.0375	0.0375	0
3	0.860662	0.860662	0
4	3.095238	3.095238	0
5	6.147593	6.198542	0.050949
6	17.416226	17.698124	0.281898
7	45.42388	46.799542	1.375662
8	130.05051	135.83888	5.78837
9	375.2722	398.37987	23.10767
10	1120.9855	1209.5911	88.6056
11	3407.667	3739.9026	332.2356
12	10562.556	11789.079	1226.523
13	33233.05	37716.241	4483.191
14	106003.64	122280.85	16277.21
15	342117.6	400957.33	58839.73

TABLE III (C): FOR THE CASE OF N=5 AND α =15.889

Moment	Base	Modified	Difference
1	0.16667	0.16667	0
2	0.01984	0.01984	0
3	1.183216	1.183216	0
4	4.2	4.2	0
5	11.264216	11.288047	0.023831
6	37.59636	37.807835	0.211475
7	128.39399	129.83641	1.44242
8	468.91698	477.53168	8.6147
9	1781.7589	1829.9506	48.1917
10	7029.1448	7288.6934	259.5486
11	28587.686	29954.61	1366.924
12	119386.001	126488.55	7102.549
13	510100.99	546713.37	36612.38
14	2223643.53	2411541	187897.47
15	9866436.27	10828723	962286.73

TABLE III (D): FOR THE CASE OF N=10 AND α = 5.809

Moment	Base	Modified	Difference
1	0.09091	0.09091	0
2	0.00689	0.00689	0
3	1.516770	1.516770	0
4	5.775824	5.775824	0
5	20.038701	20.044157	0.005456
6	83.397363	83.474215	0.076852
7	375.06336	375.8491	0.78574
8	1827.20232	1834.2379	7.03558
9	9480.0669	9539.1617	59.0948
10	51925.7092	52405.98	480.2708
11	297950.801	301789.61	3838.809
12	1780303.63	1810761.5	30457.87
13	11023008.96	11264217	241208.04
14	70438092.6	72351227	1913134.4
15	462966384.8	478195650	15229265.2

Base, Modified densities (of distributions) and their differences for different values of 'n' and ' α '.

Fig. 1 (a). For the case of n=2 and $\alpha = 77.00$.

Fig. 1 (b). For the case of n=3 and $\alpha = 37.714$.

Fig. 1 (c). For the case of n=5 and $\alpha = 15.889$.

Fig. 1 (d). For the case of n=10 and $\alpha = 5.809$.

B. Maximum Order Statistic for Rectangular Distribution

The following Table IV gives the polynomials and the corresponding ' θ ' values for the chosen values of 'n'.

TABLE IV: POLYNOMIALS AND THE CORRESPONDING ' θ ' VALUES FOR A CHOSEN VALUE OF THE 'N'

	$g(x_n)$	$\theta = \max_i g(x_n) $
2	$-0.012987 + 0.227273x - 1.212121x^2 + 2.727273x^3 - 2.727273x^4 + x^5$	0.012987
3	$-0.026515 + 0.353535x - 1.590909x^2 + 3.181818x^3 - 2.916667x^4 + x^5$	0.026515
5	$-0.062937 + 0.629371x - 2.307692x^2 + 3.956044x^3 - 3.214286x^4 + x^5$	0.062937
10	$-0.172171 + 1.291279x - 3.756449x^2 + 5.321637x^3 - 3.684211x^4 + x^5$	0.172171

Table V below gives the α for different n and θ values.

TABLE V: α VALUES FOR THE DIFFERENT 'N' AND ' θ ' VALUES

n	2	3	5	10
θ	0.012987	0.026515	0.062937	0.172171
α	77.00	37.714	15.88889	5.808192
$\frac{1}{\theta}$				

Thus, the general formula for modified maximum order statistic distribution is:

$$f_m(x) = (n \cdot f(x) (F(x))^{n-1}) \{1 + \alpha g(x_n)\}$$

Thus, for instance, when n =2, we have:

$$f_m(x) = 2 \cdot x \cdot \{1 + \alpha g(x_n)\}, \text{ and } \alpha \leq 77.00$$

When n =3, we have $f_m(x) = 3 \cdot x^2 \cdot \{1 + \alpha g(x_n)\}$, and $\alpha \leq 37.714$. When n =5, we have $f_m(x) = 5 \cdot x^4 \cdot \{1 + \alpha g(x_n)\}$, and $\alpha \leq 15.889$ and when n =10, we have $f_m(x) = 10 \cdot x^9 \cdot \{1 + \alpha g(x_n)\}$, and $\alpha \leq 5.808$. Similarly, $f_m(x)$ can be computed for any value of 'n'.

Table VI (a), Table VI (b), Table VI (c) and Table VI (d) clearly show that the first four standardized moments (first two moments are actual moments, i.e., mean and variance) are same for base and modified distributions and higher order moments are different.

Standardized Moments for base, modified distributions and their difference.

TABLE VI (A): FOR THE CASE OF N=2 AND $\alpha = 77.00$

Moment	Base	Modified	Difference
1	0.6667	0.6667	0
2	0.0557	0.0557	0
3	-0.56569	-0.56569	0
4	2.4	2.4	0
5	-3.232488	-3.149862	0.082626
6	8.857143	8.587443	-0.269700
7	-16.970563	-15.902591	1.067972
8	41.6	38.377831	-3.222169
9	-92.155298	-82.222764	9.932534
10	222.54546	194.02521	-28.52025
11	-525.6523	-444.10499	81.54731
12	1283.517	1055.8334	-227.6836
13	-3134.221	-2502.24	631.981
14	7772.8	6035.14	-1737.66
15	-19377.06	-14618.299	4758.761

TABLE VI (B): FOR THE CASE OF N=3 AND $\alpha = 37.714$

Moment	Base	Modified	Difference
1	0.75	0.75	0
2	0.0375	0.0375	0
3	-0.860663	-0.860663	0
4	3.095235	3.095235	0
5	-6.147593	-6.096628	0.050965
6	17.416226	17.134291	-0.281935
7	-45.423879	-44.048096	1.375783
8	130.05051	124.26178	-5.78873
9	-375.2723	-352.16344	23.10886
10	1120.986	1032.3761	-88.6099
11	-3407.667	-3075.4184	332.2486
12	10562.556	9335.9913	-1226.564
13	-33233.051	-28749.723	4483.328
14	106003.64	89725.97	-16277.67
15	-342117.6	-283276.27	58841.33

TABLE VI (C): FOR THE CASE OF N=5 AND $\alpha = 15.889$

Moment	Base	Modified	Difference
1	0.83333	0.83333	0
2	0.01984	0.01984	0
3	-1.18323	-1.18323	0
4	4.2	4.2	0
5	-11.264216	-11.240479	0.023737
6	37.596364	37.385182	-0.211182
7	-128.39399	-126.95269	1.4413
8	468.9169	460.30656	-8.61034
9	-1781.7589	-1733.5846	48.1743
10	7029.145	6769.6681	-259.4769
11	-28587.686	-27221.069	1366.617
12	119386.001	112284.8	-7101.201
13	-510100.99	-473494.57	36606.42
14	2223643.53	2035773	-187870.53
15	-9866436.27	-8904273.2	962163.07

TABLE VI (D): FOR THE CASE OF N=10 AND $\alpha = 5.808$

Moment	Base	Modified	Difference
1	0.90909	0.90909	0
2	0.00689	0.00689	0
3	-1.51676	-1.51676	0
4	5.77582	5.77582	0
5	-20.038701	-20.033232	0.005469
6	83.39736	83.320461	-0.076899
7	-375.06336	-374.27738	0.78598
8	1827.2023	1820.1655	-7.0368
9	-9480.0669	-9420.9659	59.101
10	51925.709	51445.403	-480.306
11	-297950.801	-294111.79	3839.011
12	1780303.63	1749844.5	-30459.13
13	-11023008.97	-10781793	241215.97
14	70438092.54	68524905	-1913187.54
15	-462966384.8	-447736760	15229624.8

Base, Modified order statistic densities (of distributions) and their differences for different values of 'n' and ' α '.

Fig. 2 (a). For the case of n=2 and $\alpha = 77.00$.

Fig. 2 (b). For the case of n=3 and $\alpha = 37.714$.

Fig. 2 (c). For the case of n=5 and $\alpha = 15.889$.

Fig. 2 (d). For the case of n=10 and $\alpha = 5.808$.

C. Exponential Distribution

In this section we consider the Exponential Distribution, defined by:

$$F(x) = 1 - e^{-x}, 0 \leq x < \infty,$$

and now the Minimum Order statistic density function of Exponential Distribution is:

$$f_n(x) = n \cdot e^{-x} \cdot (1 - (1 - e^{-x}))^{n-1}$$

and Maximum Order statistic density function of Exponential Distribution is:

$$f_n(x) = n.e^{-x}.(1 - e^{-x})^{n-1}$$

where 'n' is the sample size.

Now we consider the polynomials of order 5 as the modifier i.e., k=4, m=5 (polynomial order), one gets M, such that:

$$\alpha .M. \bar{b} = \bar{\Delta}$$

Solving the above equation for ' $\bar{b}$ ', we get the modifier polynomial coefficients as ' $\bar{b}$ ' from which we get the following 5th degree polynomial in 'x' with 'n', sample size.

1) Minimum Order Statistics for Exponential Distribution

The following Table VII gives the polynomials and the corresponding ' θ ' values for the chosen values of 'n'.

TABLE VII: POLYNOMIALS AND THE CORRESPONDING ' θ ' VALUES FOR THE CHOSEN VALUES OF 'N'

n	$g(x_n)$	$\theta = \max_x g(x_n) $
2	$(-0.740741+11.1111*x-33.3333*x^2+33.3333*x^3-12.5*x^4+1.5*x^5)*e^{-x}$	4.6471
3	$(-0.15625+3.125*x-12.5*x^2+16.66667*x^3-8.3333*x^4+\frac{4}{3}*x^5)*e^{-x}$	7.0132
5	$(-0.018519+0.55556*x-3.3333*x^2+6.66667*x^3-5*x^4+6/5*x^5)*e^{-x}$	10.35
10	$(-0.000819+0.045079*x-0.49587*x^2+1.81818*x^3-2.5*x^4+1.1*x^5)*e^{-x}$	14.452

Table VIII below gives the ' α ' for the different 'n' and ' θ ' values.

TABLE VIII: ' α ' FOR THE DIFFERENT 'N' AND ' θ ' VALUES

n	2	3	5	10
θ	4.6471	7.0132	10.35	14.452
$\alpha = \frac{1}{\theta}$	0.215188	0.142588	0.096618	0.069195

Thus, the general formula for modified minimum order statistic distribution is:

$$f_m(x) = (n.f(x)(1 - F(x))^{n-1}).\{1 + \alpha g(x_n)\}$$

Thus, for instance, when n =2, we have:

$$f_m(x) = 2.(e^{-x})^2.\{1 + \alpha g(x_n)\}, \text{ and } \alpha \leq 0.215$$

when n =3, we have:

$$f_m(x) = 3.(e^{-x})^3.\{1 + \alpha g(x_n)\}, \text{ and } \alpha \leq 0.143$$

when n =5, we have:

$$f_m(x) = 5.(e^{-x})^5.\{1 + \alpha g(x_n)\}, \text{ and } \alpha \leq 0.097$$

and, when n=10, we have:

$$f_m(x) = 10.(e^{-x})^{10}.\{1 + \alpha g(x_n)\}, \text{ and } \alpha \leq 0.069$$

Similarly, $f_m(x)$ can be computed for any value of 'n'.

Table IX (a), Table IX (b), Table IX (c) and Table IX (d) clearly show that the first four moments (first two moments are actual moments, i.e., mean and variance) are same for base and modified distributions and higher order moments are different.

Standardized moments for base, modified distributions and their difference.

TABLE IX (A): FOR THE CASE OF N=2 AND $\alpha = 0.215188$

Moment	Base	Modified	Difference
1	0.5	0.5	0
2	0.25	0.25	0
3	2	2	0
4	9	9	0
5	44	45.679281	1.679281
6	265	295.22703	30.22703
7	1854	2265.4234	411.4234
8	14833	19963.383	5130.383
9	133496	196149.89	62653.89
10	1334961	2106603.1	771642.1
11	14684570	24409962	9725392
12	176214841	302620190	126405349
13	2290792932	3992023000	1701230068
14	32071101049	55828217000	23757115951
15	481066515734	825604820000	344538304266

TABLE IX (B): FOR THE CASE OF N=3 AND $\alpha = 0.142588$

Moment	Base	Modified	Difference
1	0.33333	0.33333	0
2	0.11111	0.11111	0
3	2	2	0
4	9	9	0
5	44	44.475836	0.475836
6	265	274.99251	9.99251
7	1854	2010.1328	156.1328
8	14833	17054.665	2221.665
9	133496	164346.58	30850.58
10	1334961	1766011	431050
11	14684570	20838138	6153568
12	176214841	266702900	90488059
13	2290792932	3667416000	1376623068
14	32071101049	53786932000	21715830951
15	481066515734	836629230000	355562714266

TABLE IX (C): FOR THE CASE OF N=5 AND $\alpha = 0.096618$

Moment	Base	Modified	Difference
1	0.2	0.2	0
2	0.04	0.04	0
3	2	2	0
4	9	9	0
5	44	44.071907	0.071907
6	265	266.72577	1.72577
7	1854	1884.4527	30.4527
8	14833	15320.019	487.019
9	133496	141075.45	7579.45
10	1334961	1453431.5	118470.5
11	14684570	16574134	1889564
12	176214841	207229800	31014959
13	2290792932	2817091500	526298568
14	32071101049	41326418000	9255316951
15	481066515734	649930960000	168864444266

TABLE IX (D): FOR THE CASE OF N=10 AND $\alpha = 0.069195$

Moment	Base	Modified	Difference
1	0.1	0.1	0
2	0.01	0.01	0
3	2	2	0
4	9	9	0
5	44	44.003842	0.003842
6	265	265.10268	0.10268
7	1854	1856.0008	2.0008
8	14833	14868.211	35.211
9	133496	134097.72	601.72
10	1334961	1345274	10313
11	14684570	14864764	180194
12	176214841	179452590	3237749
13	2290792932	2350905400	60112468
14	32071101049	33227220000	1156118951
15	481066515734	504127830000	23061314266

Base, Modified densities (of distributions) and their differences for different values of 'n' and ' α '.

Fig. 3 (a). For the case of n=2 and $\alpha = 0.215188$.

Fig. 3 (b). For the case of n=3 and $\alpha = 0.142588$.

Fig. 3 (c). For the case of n=5 and $\alpha = 0.096618$.

Fig. 3 (d). For the case of n=10 and $\alpha = 0.069195$.

2) Maximum Order Statistics for Exponential Distribution

The following Table X gives the polynomials and the corresponding ' θ ' values for the chosen values of 'n'.

TABLE X: POLYNOMIALS AND THE CORRESPONDING ' θ ' VALUES FOR THE CHOSEN VALUES OF 'N'

n	$g(x_n)$	$\theta = \max_x g(x_n) $
2	$3.(-12.118258+70.011972*x-106.67163*x^2+60.315411*x^3-13.510848*x^4+x^5)*e^{-x}$	36.35438
3	$4.(-24.891753+108.11083*x-138.35147*x^2+69.608755*x^3-14.356954*x^4+x^5)*e^{-x}$	99.55202
5	$6.(-61.624876+194.79038*x-200.01646*x^2+85.819686*x^3-15.719125*x^4+x^5)*e^{-x}$	369.68577
10	$11.(-198.066606+437.644659*x-340.320795*x^2+117.346555*x^3-18.067320*x^4+x^5)*e^{-x}$	2183.40611

Table XI below gives the ' α ' for the different 'n' and ' θ ' values.

TABLE XI: ' α ' FOR THE DIFFERENT 'N' AND ' θ ' VALUES

n	2	3	5	10
θ	36.35438	99.55202	369.68577	2183.40611
$\alpha = \frac{1}{\theta}$	0.027507	0.010045	0.002705	0.000458

Thus, the general formula for modified maximum order statistic distribution is:

$$f_m(x) = (n.f(x) F(x)^{n-1}).\{1 + \alpha g(x_n)\}$$

Thus, for instance, when n =2, we have:

$$f_m(x) = 2 e^{-x} (1 - e^{-x}).\{1 + \alpha g(x_n)\}, \text{ and } \alpha \leq 0.028$$

When n =3, we have:

$$f_m(x) = 3 e^{-x} (1 - e^{-x})^2.\{1 + \alpha g(x_n)\}, \text{ and } \alpha \leq 0.010$$

When n =5, we have:

$$f_m(x) = 5 e^{-x} (1 - e^{-x})^4.\{1 + \alpha g(x_n)\}, \text{ and } \alpha \leq 0.003$$

And, when n=10, we have:

$$f_m(x) = 10 e^{-x} (1 - e^{-x})^9 \cdot \{1 + \alpha g(x_n)\}, \text{ and } \alpha \leq 0.001$$

Similarly, $f_m(x)$ can be computed for any value of 'n'.

Table XII (a), Table XII (b), Table XII (c) and Table XII (d) clearly show that the first four moments (first two moments are actual moments, i.e., mean and variance) are same for base and modified distributions and higher order moments are different.

Standardized moments for base, modified distributions and their differences are:

TABLE XII (A): FOR THE CASE OF N=2 AND $\alpha = 0.027507$

Moment	Base	Modified	Difference
1	1.5	1.5	0
2	1.25	1.25	0
3	1.6099683	1.6099683	0
4	7.08	7.08	0
5	30.267416	30.709885	0.442469
6	164.52	168.52689	4.00689
7	1028.770155	1061.2672	32.49704491
8	7367.1696	7615.731	248.5614
9	59314.19247	61222.024	1907.831527
10	530584.1885	545557.0800	14972.89152
11	5220537.447	5341970.200	121432.7527
12	56034220.46	57057003.00	1022782.539
13	651549760.6	660517250.0	8967489.408
14	8158749077	8240666500	81917423
15	109461477958	110241150000	779672042

TABLE XII (B): FOR THE CASE OF N=3 AND $\alpha = 0.010045$

Moment	Base	Modified	Difference
1	1.8333333	1.8333333	0
2	1.361111	1.361111	0
3	1.463557	1.463557	0
4	6.4810505	6.4810505	0
5	26.13220682	26.31818	0.1859732
6	137.0329115	138.47994	1.4470285
7	820.1818071	831.17275	10.990943
8	5631.675868	5710.0885	78.412632
9	43452.37273	44019.32	566.94727
10	372510.0000	376718.5100	4203.9686
11	3512543.381	3544848.300	32304.919
12	36130441.43	36388731.00	258289.57
13	402604180.1	404757060.0	2152879.9
14	4831297319	4850014700	18717381
15	62116982775	62286692000	169709225

TABLE XII (C): FOR THE CASE OF N=5 AND $\alpha = 0.002705$

Moment	Base	Modified	Difference
1	2.2833333	2.2833333	0
2	1.4636111	1.4636111	0
3	1.3392214	1.3392214	0
4	6.0259736	6.0259736	0
5	22.992498	23.052957	0.060459
6	117.26109	117.65746	0.39637
7	675.46393	678.33678	2.87285
8	4476.1153	4495.2478	19.1325
9	33303.3457	33434.589	131.2433
10	275348.172	276272.53	924.358
11	2503811.626	2510581.7	6770.074
12	24836602.29	24888285	51682.71
13	266890678	267302610	411932.03
14	3088545498	3091974100	3428602.4
15	38294386225	38324173000	29786775

TABLE XII (D): FOR THE CASE OF N=10 AND $\alpha = 0.000458$

Moment	Base	Modified	Difference
1	2.9289684	2.9289684	0
2	1.5497678	1.5497678	0
3	1.2414152	1.2414152	0
4	5.7030856	5.7030856	0
5	20.737247	20.749936	0.012689
6	103.75549	103.82458	0.06909
7	579.26026	579.75352	0.49326
8	3734.1203	3737.1865	3.0662
9	26995.359	27015.562	20.203
10	216920.36	217056.47	136.11
11	1916913.2	1917871.4	958.2
12	18478948.5	18485987	7038.5
13	192974445.5	193028510	54064.53
14	2170205882	2170639800	433918.04
15	26149420508	26153057000	3636492.3

Base, Modified densities (of distributions) and their differences for different values of 'n' and ' α '.

Fig. 4 (a). For the case of n=2 and $\alpha = 0.027507$.

Fig. 4 (b). For the case of n=3 and $\alpha = 0.010045$.

Fig. 4 (c). For the case of n=5 and $\alpha = 0.002705$.

Fig. 4 (d). For the case of n=10 and $\alpha = 0.000458$.

D. Gamma Distribution

Here we consider the Gamma Distribution, defined by:

$$f(x) = \frac{1}{[b\Gamma(c)]} \left[\frac{x}{b}\right]^{c-1} e^{-\frac{x}{b}} \text{ for } 0 \leq x < \infty \text{ and } c > 0.$$

$$F(x) = 1 - (1+x)e^{-x}, 0 \leq x < \infty,$$

where shape parameter(c)=2 and scale parameter(b)=1.

Minimum Order statistic distribution function of Gamma Distribution is:

$$f_n(x) = n.x.e^{-x} \cdot ((1+x)e^{-x})^{n-1}$$

and Maximum Order statistic distribution function of Gamma Distribution is:

$$f_n(x) = n.x.e^{-x} \cdot (1 - (1+x)e^{-x})^{n-1}$$

where 'n' is the sample size.

Now we consider the polynomials of order 5 as the modifier i.e. k=4, m=5(polynomial order), one gets M, such that:

$$\alpha \cdot M. \bar{b} = \bar{\Delta}$$

Solving the above equation for ' $\bar{b}$ ', we get the modifier polynomial coefficients as ' $\bar{b}$ ', from which we get the following 5th degree polynomial in 'x' with 'n', sample size.

1) Minimum Order Statistics for Gamma Distribution

The following Table XIII gives the polynomials and the corresponding ' θ ' values for the chosen values of 'n'.

Table XIV below gives the ' α ' for the different 'n' and ' θ ' values.

Thus, the general formula for modified minimum order statistic distribution is:

$$f_m(x) = (n.f(x)(1-F(x))^{n-1}) \cdot \{1 + \alpha g(xn)\}$$

Thus, for instance, when n=2, we have:

$$fm(x) = 2.x.(1+x).(e-x)^2.\{1 + \alpha g(xn)\}, \text{ and } \alpha \leq 0.072$$

When n=3, we have:

$$fm(x) = 3.x.(1+x)^2.(e-x)^3.\{1 + \alpha g(xn)\}, \text{ and } \alpha \leq 0.151$$

When n=5, we have:

$$fm(x) = 4.x.(1+x)^4.(e-x)^4.\{1 + \alpha g(xn)\}, \text{ and } \alpha \leq 0.105$$

And, when n=10, we have:

$$fm(x) = 10.x.(1+x)^9.(e-x)^{10}.\{1 + \alpha g(xn)\}, \text{ and } \alpha \leq 0.076$$

TABLE XIII: POLYNOMIALS AND THE CORRESPONDING ' θ ' VALUES FOR THE CHOSEN VALUES OF 'N'

n	$g(x_n)$	$\theta = \max_x g(x_n) $
2	$(-13.992850+90.939227*x-160.83198*x^2+108.25154*x^3-29.47741*x^4+2.7*x^5)*e^{-x}$	13.99286
3	$(-4.234567+33.041736*x-71.020412*x^2+58.630829*x^3-19.725296*x^4+128/57*x^5)*e^{-x}$	6.627652
5	$(-0.880615+8.84918*x-24.85252*x^2+27.10948*x^3-12.1614*x^4+1944/1045*x^5)*e^{-x}$	9.513843
10	$(-0.10085+1.46378*x-6.03795*x^2+9.8036*x^3-6.61987*x^4+928/603*x^5)*e^{-x}$	13.13267

TABLE XIV: ' α ' FOR THE DIFFERENT 'N' AND ' θ ' VALUES

n	2	3	5	10
θ	13.99286	6.627652	9.513843	13.13267
$\alpha = \frac{1}{\theta}$	0.07147	0.15088	0.10511	0.07615

Similarly, fm(x) can be computed for any value of 'n'.

Table XV (a), Table XV (b), Table XV (c) and Table XV (d) clearly show that the first four moments are same for base and modified distributions and higher order moments are different.

Standardized moments for base, modified distributions and their difference.

TABLE XV (A): FOR THE CASE OF N=2 AND $\alpha = 0.072$

Moment	Base	Modified	Difference
1	1.25	1.25	0
2	0.6875	0.6875	0
3	1.260866	1.260866	0
4	5.330579	5.330579	0
5	18.290027	18.943591	0.653564
6	83.076634	89.101117	6.024483
7	417.01669	466.74904	49.73235
8	2377.01421	2760.4959	383.48169
9	15015.669	17965.593	2949.924
10	104285.556	127350.3	23064.744
11	789105.27	974504.23	185398.96
12	6460188.8	8000625.5	1540436.7
13	56879071.2	70146228	13267156.8
14	535843756.8	654434810	118591053.2
15	5377495920	6478080300	1100584380

TABLE XV (B): FOR THE CASE OF N=3 AND $\alpha = 0.151$

Moment	Base	Modified	Difference
1	0.9629628	0.9629628	0
2	0.38134424	0.38134424	0
3	1.176221	1.176221	0
4	4.98639	4.98639	0
5	16.134377	16.8228	0.688423
6	70.52204	77.57436	7.05232
7	335.6914	398.18257	62.49117
8	1814.3911	2329.3622	514.9711
9	10822.144	15029.027	4206.883
10	70787.71	105583.05	34795.34
11	503159.22	798066.49	294907.27
12	3860720.03	6437452.8	2576732.77
13	31792784.9	55076923	23284138.1
14	279607410	497543700	217936290
15	2615021648	4729167300	2114145652

TABLE XV (C): FOR THE CASE OF N=5 AND $\alpha = 0.105$

Moment	Base	Modified	Difference
1	0.70208	0.70208	0
2	0.18791567	0.18791567	0
3	1.078873	1.078873	0
4	4.613901	4.613901	0
5	13.860755	14.00737	0.146615
6	57.970922	59.565719	1.594797
7	258.32116	273.01114	14.68998
8	1308.6773	1433.8007	125.1234
9	7270.795	8321.3844	1050.5894
10	44154.443	53048.397	8893.954
11	290327.63	367206.64	76879.01
12	2054238.799	2737203.3	682964.501
13	15553883.1	21811461	6257577.905
14	125434973.7	184676320	59241346.3
15	1073059488	1653035100	579975612

TABLE XV (D): FOR THE CASE OF N=10 AND $\alpha = 0.076$

Moment	Base	Modified	Difference
1	0.46602157	0.46602157	0
2	0.076028	0.076028	0
3	0.966763	0.966763	0
4	4.217005	4.217005	0
5	11.506425	11.521712	0.015287
6	45.82875	45.995427	0.166677
7	187.85691	189.38412	1.52721
8	879.62922	892.48367	12.85445
9	4472.8236	4578.9275	106.1039
10	24762.446	25641.319	878.873
11	147654.587	155056.45	7401.863
12	943290.544	1007110.7	63820.156
13	6421606.76	6987126.9	565520.14
14	46381439.91	51542227	5160787.09
15	354066308	402618920	48552612

Base, Modified densities (of distributions) and their differences for different values of 'n' and ' α '.

Fig. 5 (a). For the case of n=2 and $\alpha = 0.072$.

Fig. 5 (b). For the case of n=3 and $\alpha = 0.151$.

Fig. 5 (c). For the case of n=5 and $\alpha = 0.105$.

Fig. 5 (d). For the case of n=10 and $\alpha = 0.076$.

2) Maximum Order Statistics for Gamma Distribution

The following Table XVI gives the polynomials and the corresponding ' θ ' values for the chosen values of 'n'.

TABLE XVI: POLYNOMIALS AND THE CORRESPONDING ' θ ' VALUES FOR THE CHOSEN VALUES OF 'N'

n	$g(x_n)$	$\theta = \max_x g(x_n) $
2	$(-643.09026+1972.0058*x-1916.1244*x^2+768.20953*x^3-130.73209*x^4+54/7*x^5)*e^{-x}$	643.0868
3	$(-2114.6917+5072.1983*x-4108.7548*x^2+1434.1843*x^3-219.14792*x^4+1152/97*x^5)*e^{-x}$	2114.1649
5	$(-9322.2623+17118.467*x-11202.126*x^2+3291.2109*x^3-437.05601*x^4+3673/174*x^5)*e^{-x}$	9345.794
10	$(-63276.988+86530.723*x-44013.615*x^2+10424.002*x^3-1151.0504*x^4+4747/100*x^5)*e^{-x}$	62500

Table XVII below gives the ‘ α ’ for the different ‘n’ and ‘ θ ’ values.

TABLE XVII: ‘ α ’ FOR THE DIFFERENT ‘N’ AND ‘ θ ’ VALUES				
n	2	3	5	10
θ	643.086 8	2114.1649	9345.794	62500
$\alpha = \frac{1}{\theta}$	0.00155 5	0.000473	0.000107	0.000016

Thus, the general formula for modified minimum order statistic distribution is:

$$f_m(x) = (n.f(x) F(x)^{n-1}).\{1 + \alpha g(x_n)\}$$

Thus, for instance, when n=2, we have:

$$f_m(x) = x.(2-2.(1+x). e^{-x}).e^{-x}.\{1 + \alpha g(x_n)\}, \text{ and } \alpha \leq 0.0016$$

When n=3, we have:

$$f_m(x) = 3.x.(2-2.(1+x). e^{-x})^2.e^{-x}.\{1 + \alpha g(x_n)\}$$

and $\alpha \leq 0.0005$.

When n=5, we have:

$$f_m(x) = 5.x.(2-2.(1+x). e^{-x})^4.e^{-x}.\{1 + \alpha g(x_n)\},$$

and $\alpha \leq 0.0001/$

And, when n=10, we have:

$$f_m(x) = 10.x.(2-2.(1+x). e^{-x})^9.e^{-x}.\{1 + \alpha g(x_n)\},$$

and $\alpha \leq 0.00002$.

Similarly, $f_m(x)$ can be computed for any value of ‘n’.

Table XVIII (a), Table XVIII (b), Table XVIII (c) and Table XVIII (d) clearly show that the first four moments are same for base and modified distributions and higher order moments are different.

Standardized moments for base, modified distributions and their difference.

TABLE XVIII (A): FOR THE CASE OF N=2 AND $\alpha = 0.0016$

Moment	Base	Modified	Difference
1	2.75	2.75	0
2	2.1875	2.1875	0
3	1.207363	1.207363	0
4	5.346123	5.346123	0
5	18.369031	18.403152	0.034121
6	86.194169	86.348606	0.154437
7	446.0136	447.00664	0.99304
8	2644.81	2650.1859	5.3759
9	17459.6821	17490.941	31.2589
10	127343.1851	127527.99	184.8049
11	1016041.658	1017184	1142.342
12	8802954.783	8810310.8	7356.017
13	82289021.72	82338496	49474.277
14	825438593.9	825785990	347396.1
15	8843163318	8845708300	2544982.1

TABLE XVIII (B): FOR THE CASE OF N=3 AND $\alpha = 0.0005$

Moment	Base	Modified	Difference
1	3.2129629	3.2129629	0
2	2.2355109	2.2355109	0
3	1.138663	1.138663	0
4	5.168295	5.168295	0
5	17.154259	17.167078	0.012819
6	79.991832	80.040186	0.048354
7	406.573367	406.89741	0.324043
8	2379.5459	2381.1881	1.6422
9	15489.83831	15499.271	9.43269
10	111486.559	111540.71	54.151
11	877988.787	878317.02	328.233
12	7510346.957	7512417.4	2070.443
13	69329994.62	69343668	13673.38
14	686899076.4	686993440	94363.57
15	7269644378	7270324800	680422.02

TABLE XVIII (C): FOR THE CASE OF N=5 AND $\alpha = 0.0001$

Moment	Base	Modified	Difference
1	3.8082722	3.8082722	0
2	2.2525964	2.2525964	0
3	1.087835	1.087835	0
4	5.056374	5.056374	0
5	16.345891	16.349486	0.003595
6	76.167268	76.178468	0.0112
7	382.66238	382.74398	0.0816
8	2224.767	2225.1507	0.3837
9	14372.845	14375.065	2.22
10	102755.895	102768.3	12.405
11	804026.319	804100.77	74.451
12	6835741.564	6836204.3	462.736
13	62733720.55	62736740	3019.45
14	618049268.7	618069850	20581.3
15	6505409604	6505556200	146595.82

TABLE XVIII (D): FOR THE CASE OF N=10 AND $\alpha = 0.00002$

Moment	Base	Modified	Difference
1	4.622957	4.622957	0
2	2.2313893	2.2313893	0
3	1.058087	1.058087	0
4	5.010669	5.010669	0
5	15.969113	15.969751	0.000638
6	74.717092	74.718782	0.00169
7	374.102919	374.11657	0.013651
8	2176.49748	2176.5576	0.06012
9	14063.289	14063.646	0.357
10	100650.878	100652.84	1.962
11	788672.57	788684.42	11.85
12	6717437.33	6717511.1	73.77
13	61779618.47	61780104	485.53
14	610113872	610117230	3358.04
15	6438810173	6438834700	24527.1

Base, Modified densities (of distributions) and their differences for different values of ‘n’ and ‘ α ’.

Fig. 6(a). For the case of n=2 and $\alpha = 0.0016$.

Fig. 6 (b). For the case of $n=3$ and $\alpha = 0.0005$.

Fig. 6 (c). For the case of $n=5$ and $\alpha = 0.0001$.

Fig. 6 (d). For the case of $n=10$ and $\alpha = 0.00002$.

E. Weibull Distribution

Here we consider Weibull Distribution, defined by:

$$f(x) = \alpha \cdot \beta \cdot x^{\alpha-1} \cdot e^{-\beta \cdot x^\alpha}, 0 \leq x \leq \infty, \alpha > 0, \beta > 0$$

$$F(x) = 1 - e^{-x^\alpha}, 0 \leq x \leq \infty, \text{ where } \alpha=2 \text{ and } \beta=3$$

Minimum Order statistic distribution function of Weibull Distribution is:

$$f_n(x) = n \cdot 6 \cdot x \cdot e^{-3x^2} \cdot (e^{-3x^2})^{n-1}$$

Maximum Order statistic distribution function of Weibull

Distribution is:

$$f_n(x) = n \cdot 6 \cdot x \cdot e^{-3x^2} \cdot (1 - e^{-3x^2})^{n-1}$$

where 'n' is the sample size.

Now we consider the polynomials of order 5 as the modifier i.e. $k=4, m=5$ (polynomial order), one gets M, such that:

$$\alpha \cdot M \cdot \bar{b} = \bar{\Delta}$$

Solving the above equation for ' $\bar{b}$ ', we get the modifier polynomial coefficients as ' $\bar{b}$ ' from which we get the following 5th degree polynomial in 'x' with 'n', sample size.

1) Minimum Order Statistics for Weibull Distribution

The following Table XIX gives the polynomials and the corresponding ' θ ' values for chosen values of the 'n'.

TABLE XIX: POLYNOMIALS AND THE CORRESPONDING ' θ ' VALUES FOR THE CHOSEN VALUES OF 'N'

n	$g(x_n)$	$\theta = \max_x g(x_n) $
2	$(-0.030903 + 0.635727 \cdot x - 3.845494 \cdot x^2 + 9.440648 \cdot x^3 - 9.911061 \cdot x^4 + 419/114 \cdot x^5) \cdot e^{-x}$	44.51171
3	$(-0.011459 + 0.284615 \cdot x - 2.082356 \cdot x^2 + 6.191001 \cdot x^3 - 7.878382 \cdot x^4 + 4547/1283 \cdot x^5) \cdot e^{-x}$	47.39561
5	$(-0.003265 + 0.103195 \cdot x - 0.962506 \cdot x^2 + 3.652645 \cdot x^3 - 5.938677 \cdot x^4 + 3706/1085 \cdot x^5) \cdot e^{-x}$	50.51015
10	$(-0.0005899 + 0.025979 \cdot x - 0.338337 \cdot x^2 + 1.795094 \cdot x^3 - 4.084315 \cdot x^4 + 2826/859 \cdot x^5) \cdot e^{-x}$	53.88802

Table XX below gives the ' α ' for the different 'n' and ' θ ' values.

TABLE XX: ' α ' FOR THE DIFFERENT 'N' AND ' θ ' VALUES

n	2	3	5	10
θ	44.51171	47.39561	50.51015	53.88802
$\alpha = \frac{1}{\theta}$	0.022466	0.021099	0.019798	0.018557

Thus, the general formula for modified minimum order statistic distribution is:

$$f_m(x) = (n \cdot f(x) \cdot (1 - F(x))^{n-1}) \cdot \{1 + \alpha g(x_n)\}$$

Thus, for instance, when $n=2$, we have:

$$f_m(x) = 12 \cdot x \cdot (e^{-3x^2})^2 \cdot \{1 + \alpha g(x_n)\}, \text{ and } \alpha \leq 0.023$$

When $n=3$, we have:

$$f_m(x) = 18 \cdot x \cdot (e^{-3x^2})^3 \cdot \{1 + \alpha g(x_n)\}, \text{ and } \alpha \leq 0.021$$

When $n=5$, we have:

$$f_m(x) = 30.x.(e^{-3x^2})^5 .\{1 + \alpha g(x_n)\}, \text{ and } \alpha \leq 0.020$$

And, when $n=10$, we have:

$$f_m(x) = 602.x.(e^{-3x^2})^{10} .\{1 + \alpha g(x_n)\}, \text{ and } \alpha \leq 0.019$$

Similarly, $f_m(x)$ can be computed for any value of 'n'.

Table XXI (a), Table XXI (b), Table XXI (c) and Table XXI (d) clearly show that the first four moments are same for base and modified distributions and higher order moments are different.

Standardized moments for base, modified distributions, and their difference.

TABLE XXI (A): FOR THE CASE OF $N=2$ AND $\alpha = 0.023$

Moment	Base	Modified	Difference
1	0.36180063	0.36180063	0
2	0.035766973	0.035766973	0
3	0.63111061	0.63111061	0
4	3.2450892	3.2450892	0
5	5.9979693	5.9986067	0.0006374
6	21.7910581	21.795496	0.0044379
7	64.8165705	64.846631	0.0300605
8	238.8265889	239.00658	0.1799911
9	888.0432285	889.10556	1.0623315
10	3603.816332	3610.0202	6.203868
11	15301.3977	15337.859	36.4613
12	68606.16758	68822.881	216.713422
13	321656.8577	322964.68	1307.82235
14	1574474.161	1582506.7	8032.539
15	8015399.769	8065639	50239.231

TABLE XXI (B): FOR THE CASE OF $N=3$ AND $\alpha = 0.021$

Moment	Base	Modified	Difference
1	0.295408978	0.29540898	0
2	0.023844649	0.023844649	0
3	0.63111066	0.63111066	0
4	3.2450893	3.2450893	0
5	5.9979693	5.9982109	0.0002416
6	21.791058	21.792782	0.001724
7	64.816570	64.828373	0.011803
8	238.8265889	238.89811	0.0715211
9	888.0432285	888.46934	0.4261115
10	3603.816332	3606.3275	2.511168
11	15301.3977	15316.28	14.8823
12	68606.16758	68695.309	89.14142
13	321656.8576	322198.95	542.0924
14	1574474.161	1577827.2	3353.039
15	8015399.769	8036521	21121.231

TABLE XXI (C): FOR THE CASE OF $N=5$ AND $\alpha = 0.020$

Moment	Base	Modified	Difference
1	0.22882281	0.22882281	0
2	0.014306789	0.014306789	0
3	0.63111062	0.63111062	0
4	3.2450893	3.2450893	0
5	5.9979693	5.99804	0.0000707
6	21.791058	21.791574	0.000516
7	64.816570	64.820136	0.003566
8	238.8265889	238.84844	0.0218511
9	888.0432285	888.17468	0.1314515
10	3603.816332	3604.5978	0.781468
11	15301.3977	15306.069	4.6713
12	68606.16758	68634.375	28.20742
13	321656.8576	321829.5	172.6424
14	1574474.161	1575550.8	1076.639
15	8015399.769	8022225.5	6825.731

TABLE XXI (D): FOR THE CASE OF $N=10$ AND $\alpha = 0.019$

Moment	Base	Modified	Difference
1	0.16180216	0.16180216	0
2	0.007153395	0.007153395	0
3	0.63111063	0.63111063	0
4	3.2450892	3.2450892	0
5	5.9979693	5.9979822	0.0000129
6	21.791058	21.791155	0.000097
7	64.816570	64.817251	0.000681
8	238.8265889	238.83082	0.0042311
9	888.0432285	888.0689	0.0256715
10	3603.816332	3603.9704	0.154068
11	15301.3977	15302.326	0.9283
12	68606.16758	68611.82	5.65242
13	321656.8576	321691.7	34.8424
14	1574474.161	1574693	218.839
15	8015399.769	8016799	1399.231

Base, Modified densities (of distributions) and their differences for different values of 'n' and ' α '.

Fig. 7 (a). For the case of $n=2$ and $\alpha = 0.023$.

Fig. 7 (b). For the case of $n=3$ and $\alpha = 0.021$.

Fig. 7 (c). For the case of $n=5$ and $\alpha = 0.020$.

Fig. 7 (d). For the case of $n=10$ and $\alpha = 0.019$.

2) Maximum Order Statistics for Weibull Distribution

The following Table XXII gives the polynomials and the corresponding ' θ ' values for the chosen values of ' n '.

TABLE XXII: POLYNOMIALS AND THE CORRESPONDING ' θ ' VALUES FOR THE CHOSEN VALUES OF ' N '

n	$g(x_n)$	$\theta = \max_x g(x_n) $
2	$(-0.260100+2.5159902*x-8.308858*x^2+12.076373*x^3-7.888098*x^4+2458/1307*x^5)*e^{-x}$	16.6901548
3	$(-0.5430919+4.279469*x-12.116393*x^2+15.629332*x^3-9.285701*x^4+1548/755*x^5)*e^{-x}$	17.0425917
5	$(-1.241299+7.955042*x-19.044230*x^2+21.397410*x^3-11.332383*x^4+2365/1041*x^5)*e^{-x}$	17.2598622
10	$(-3.133745+16.329680*x-32.667656*x^2+31.392745*x^3-14.505414*x^4+5729/2220*x^5)*e^{-x}$	17.3183898

Table XXIII below gives the ' α ' for the different ' n ' and ' θ ' values.

TABLE XXIII: ' α ' FOR THE DIFFERENT ' N ' AND ' θ ' VALUES

n	2	3	5	10
θ	16.6901548	17.0425917	17.2598622	17.3183898
$\alpha = \frac{1}{\theta}$	0.059916	0.058677	0.057938	0.057742

Thus, the general formula for modified distributions is:

$$f_m(x) = (n.f(x) F(x)^{n-1}).\{1+\alpha g(x_n)\}$$

Thus, for instance, when $n=2$, we have:

$$f_m(x) = f_m(x) = (n.f(x) F(x)^{n-1}).\{1+\alpha g(x_n)\},$$

and $\alpha \leq 0.060$.

When $n=3$, we have:

$$f_m(x) = 18.x.(1-e^{-3x^2})^2.e^{-3x^2}.\{1+\alpha g(x_n)\},$$

and $\alpha \leq 0.059$.

When $n=5$, we have:

$$f_m(x) = 30.x.(1-e^{-3x^2})^4.e^{-3x^2}.\{1+\alpha g(x_n)\},$$

and $\alpha \leq 0.058$.

And, when $n=10$, we have:

$$f_m(x) = 60.x.(1-e^{-3x^2})^9.e^{-3x^2}.\{1+\alpha g(x_n)\}, \text{ and } \alpha \leq 0.058$$

Similarly, $f_m(x)$ can be computed for any value of ' n '.

Table XXIV (a), Table XXIV (b), Table XXIV (c) and Table XXIV (d) clearly show that the first four moments are same for base and modified distributions and higher order moments are different.

Standardized moments for base, modified distributions and their difference.

TABLE XXIV (A): FOR THE CASE OF $N=2$ AND $\alpha = 0.060$

Moment	Base	Modified	Difference
1	0.6615261	0.6615261	0
2	0.0623832	0.0623832	0
3	0.50824	0.50824	0
4	3.2471	3.2471	0
5	5.117408	5.121158	0.003750
6	21.093347	21.113969	0.020622
7	57.996152	58.13818	0.142028
8	225.90747	226.7019	0.79443
9	822.43254	827.069	4.63646
10	3424.688	3451.1025	26.4145
11	14614.08	14768.2	154.12
12	66773.27	67682	908.73
13	317926.7	323410	5483.3
14	1587282.4	1621000	33717.6
15	8246586.9	8460000	213413.1

TABLE XXIV (B): FOR THE CASE OF $N=3$ AND $\alpha = 0.059$

Moment	Base	Modified	Difference
1	0.74499716	0.74499716	0
2	0.05609035	0.05609035	0
3	0.49149336	0.49149336	0
4	3.3053239	3.3053239	0
5	5.095844	5.09962	0.003776
6	21.942139	21.962075	0.019936
7	60.007795	60.151	0.143205
8	241.61746	242.41923	0.80177
9	888.24021	893.05	4.80979
10	3800.2212	3828.079	27.8578
11	16528.233	16696.08	167.847
12	77478.224	78480	1001.776
13	377804.73	383937	6132.27
14	1936593.2	1975800	39206.8
15	10331890.7	10588000	256109.3

TABLE XXIV (C): FOR THE CASE OF $N=5$ AND $\alpha = 0.058$

Moment	Base	Modified	Difference
1	0.84406467	0.84406467	0
2	0.04866594	0.04866594	0
3	0.504028	0.504028	0
4	3.3926232	3.3926232	0
5	5.38216629	5.385652	0.00348571
6	23.57669943	23.59536	0.01866057
7	66.42705876	66.5689	0.14184124
8	274.9432889	275.751	0.8077111
9	1043.443504	1048.603	5.159496
10	4614.875276	4644	29.124724
11	20781.00683	20966.8	185.79317
12	101035.2443	102362.32	1327.0757
13	511516.7873	520710	9193.2127
14	2726030.672	2775600	49569.328
15	15134717.87	15595236	460518.13

TABLE XXIV (D): FOR THE CASE OF $N=10$ AND $\alpha = 0.058$

Moment	Base	Modified	Difference
1	0.96747966	0.96747966	0
2	0.040305855	0.040305855	0
3	0.54918816	0.54918816	0
4	3.5180255	3.5180255	0
5	6.064009739	6.0668884	0.002878661
6	26.30380519	26.3178	0.01399481
7	79.40665313	79.53	0.12334687
8	336.1194064	336.9	0.7805936
9	1351.259729	1357	5.740271
10	6224.354778	6262.7599	38.405122
11	29575.4057	29809.424	234.0183
12	151094.7793	155648.99	4554.2107
13	807579.7406	838170.41	30590.6694
14	4543038.1	4449212.3	-93825.8
15	26678741.63	30000000	3321258.37

Base, Modified densities (of distributions) and their differences for different values of 'n' and ' α '.

Fig. 8 (a). For the case of $n=2$ and $\alpha = 0.060$.

Fig. 8 (b). For the case of $n=3$ and $\alpha = 0.059$.

Fig. 8 (c). For the case of $n=5$ and $\alpha = 0.058$.

Fig. 8 (d). For the case of $n=10$ and $\alpha = 0.058$.

IV. CONCLUSION

It is evident from the above tables and figures that as 'n' increase for Rectangular/ Exponential distributions the higher order moment's differences are becoming bigger for both minimum and maximum order statistic distributions.

As expected, the higher order moments for the distributions with different 'n' are significantly different from one another and the shape of the corresponding density functions shows considerable wide pattern.

For instance, in the case of Rectangular and Exponential distributions the shape of the curves is similar for $n=5$ and 10 (Fig. 1 c and Fig. 1 d), but the higher order moments differences are quite significantly different large in maximum and minimum order statistic. Whereas for Gamma and Weibull distributions the shape of the curve is almost similar as 'n' increases and the moment differences are low.

The present problem opens up the possibility of the alternative approach to curve fitting [9], [10] using the statistical concepts of moments and choice of base distributions based upon the first few moments while modifiers are suitably chosen to get a good approximation to data. Unlike the least squares or the spline type of approximation the approach is to choose the higher moments to tally with the observed moments instead of the base distribution moments. The important question of choosing the parameters directly linked with the errors in fitting, like

minimizing the maximum of absolute difference between the fitted function and data is yet to be explored. Since the base distributions themselves can be of varied shapes this approach should prove more satisfactory than the usual methods of curve fitting by choosing an overall function model.

One such situation, particularly of bivariate functions requiring such functions is in the area of image processing where the image can be thought of as a bivariate function with many peaks and troughs; it is difficult to envisage any simple function to approximate the image [2], [3] and piecewise local density estimation methods often are used. Instead, the present approach promises to give a more satisfactory approximation to reproduce the image (gray level frequency function) [11], [12], which is smooth and in explicit functional form.

REFERENCES

- [1] Cramer, "Mathematical methods of Statistics", Princeton University, and Princeton University Press, 1946.
- [2] Hu, M.K., "Visual Pattern Recognition by moment invariants" *IRE Trans. Info. Theory*, 1962, Volume IT-8, pp. 178-187.
- [3] Dutt, V.A.K. "Multivariate and related statistical methods in pattern recognition". Un-Published Ph.D. Thesis-Osmania University, Hyderabad, 1995.
- [4] Hara Gopal, V. V., Sampath Kumar, K. and Pandit, S.N.N, "On distributions, which differ only in higher order moments". *The Aligarh Journal of Statistics*, Vol. 24, pp. 121-141, 2004.
- [5] Hara Gopal, V. V., Sampath Kumar, K. and Pandit, S.N.N, "A class of Bivariate Distributions, which differ only in higher moments" - *Contributions to Applied and Mathematical Statistics (CAMS Guwahati) Journal* Vol. 2, pp. 32-42, March 2004(a).
- [6] Hara Gopal, V. V., Sampath Kumar, K. and Pandit, S.N.N, "A class of Distributions differing in Higher order moments" - *Proceedings of A.P. Akademi of Sciences Journal* Vol. 8, No. 2, pp. 133-138, March 2004(b).
- [7] Hara Gopal, V. V., Sampath Kumar, K. and Pandit, S.N.N" Qualitatively different distributions with same first few moments: A case study with 't' distribution", *IJOMAS Journal* Vol. 20, No. 2, pp-157-166, (March 2004(c)).
- [8] Prokop, R. J., and Reeves, A.P. "A survey of moment-based techniques for un-occluded object representation and recognition", *CVGPR: Graphical Models and Image Processing* 54(5), 438-460, 1992.
- [9] Ramberg, J.S., Dudewicz, E.J., Tadikamalla, P.R., Mykytka, E.F:" A Probability distribution and its uses in fitting Data", *Techno metrics*, Vol.21, and No.2, 1979.
- [10] Sampath Kumar, K.: "On Distributions Which differs only in higher order moments", M.Phil. Thesis, Osmania University, Hyderabad, 2001.
- [11] Teague, M.R, "Image Analysis via the General Theory of Moments" *J. Optical Soc. Of Am.*, vol. 70, pp. 920-930, Aug. 1980.
- [12] The, C. H and Chin, R. T: "On Image Analysis by the Method of Moments", *IEEE Trans, Pattern analysis and Machine Intelligence*, vol. 10, pp. 496-513, 1998.
- [13] Tukey, J. W. "The Practical Relationship between the common transformations of percentages of Counts and of amounts", Technical Report 36, *Statistical Techniques Research Group*, Princeton University, 1960.