A Numerical Method for Fredholm Integro-Differential Equation using Positive Definite Radial Kernels and A Study of the Effect of The Shape Parameter

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Abstract — **In this paper, we use two radial kernels, the Generalized Multiquadrics and the linear Laguerre-Gaussians for the formulation of radial kernel collocation method for solving problems involving Fredholm integro-differential equations. The effect of the shape parameter contained in each of the kernels, on the accuracy of the method is investigated. The method is demonstrated using two examples the numerical results displayed in form of tables and graphs. MATLAB 2018a was used for the implementation.**

***Keywords*—Integro-differential equation; radial kernel; Gauss-Legendre quadrature formula; collocation method.**

I. INTRODUCTION

A considerable number of science and engineering problems are modelled inform of integro-differential equations. According to [1], they arise naturally in a variety of physical problems in different areas of sciences such as biological science, physics, applied mathematics, engineering and other fields such as theory of elasticity, econometrics, biomechanics, heat and mass transfer, electromagnetic, electro-dynamics, fluid dynamics, oscillating magnetic field. This category of equations is in many situations very complicated to be solved exactly, since, the solution cannot be exhibited in a closed form, though it might exist [2]. Thus, finding either the analytical approximation or numerical solution of these equations is of desirable interest.

A vast range of problems has been successfully solved by the using different approaches such as can be found in [1], they solve examples to demonstrate the effectiveness of their method for the solution of linear and nonlinear Fredholm IDEs. However, there is no prescribed guarantee for the standard formulation that the collocation matrix will be non-singular [3].

Several other collocation methods have been developed to solve these type of integral equations. These methods have been based on several basis functions like Chebyshev polynomials and cubic splines and have yielded results with various degrees of accuracy. For example, see [2], [4].

Radial kernels on the other hand are powerful tools for multivariate interpolation. As noted in [5], radial basis function interpolants have the nice property of being invariant under all Euclidean transformations and this property is desirable in many applications. RBFs have been used extensively in the numerical solution of partial differential equations, mathematical finance and optimization amongst other areas [4].

The rest of the paper, we present radial kernel collocation method for solving problems involving Fredholm integro-differential equations and illustrate the efficacy of our method with two examples. We also show the relationship of the error bound with respect to the shape parameter and condition number.

II. METHOD

A. Preliminaries

According to [6], Volterra studied the hereditary influences as he was investigating a population growth model. The research resulted in a set of equations, where both differential operators and the integral

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operators appeared together in a single equation. The strange type of equations was called integro-differential equations and precisely Fredholm integro-differential equation if the limits of integration are all constant [3]. A kth order linear Fredholm integro-differential equation is given as

$$\sum_{n=0}^{k} P_n(x)u^{(n)}(x) = f(x) + \lambda \int_a^b k(x,t)\, u(t)dt, \tag{1}$$

where the initial conditions $u^{(n)}(0) = b_n,\ n = 0, 1, 2, \dots, k-1$.

and $u^{(i)}(x) = \frac{d^i u}{dx^i}$. Since the resulted equations in (1) contains both the differential operator and the integral operator, and we wish to determine the particular solution $u(x)$ of the Fredholm integro-differential equation (1). it is important first to define the initial conditions $u(0),\ u'(0),\ \dots,\ u^{(k-1)}(0)$. Any Fredholm integro-differential equation is characterized by the existence of one or more of the derivatives $u'(x), u''(x), \dots$ outside the integral sign [7], [8].

The radial kernels used in this work and their derivatives are defined as follows [5]:

Generalized Multiquadric

$$\varphi(r) = ((\varepsilon r)^2 + 1)^{5/2}$$
$$\varphi'(r) = 5\varepsilon^2 r((\varepsilon r)^2 + 1)^{3/2}$$
$$\varphi''(r) = 5\varepsilon^2((\varepsilon r)^2 + 1)^{\frac{1}{2}}(3(\varepsilon x)^2 + (\varepsilon r)^2 + 1)$$

Linear Laguerre Gaussian

$$\varphi(r) = exp(-(\varepsilon r)^2)(2 - (\varepsilon r)^2)$$
$$\varphi'(r) = 2\varepsilon^2 r\, exp(-(\varepsilon r)^2)((\varepsilon r)^2 - 3)$$
$$\varphi''(r) = 2\varepsilon^2\, exp(-(\varepsilon r)^2)((\varepsilon r)^4 - 9(\varepsilon r)^2 + 3).$$

B. Interpolation by Radial Kernels

Radial kernel interpolation is an advanced method in approximation theory for the construction of higher order accurate interpolants for scattered data up to higher dimensional spaces. The interpolation takes the form of a weighted sum of radial kernels. It is a mesh free method, meaning that the nodes need not lie on a structured grid, and does not require the formation of a mesh. It is often spectrally accurate and stable for large numbers of data nodes even in higher dimensions [5].

Given a data vector $\boldsymbol{f} = (f(x_1), f(x_2), \cdots, f(x_N))^T \in \mathbb{R}^N$ of function values obtained from some function $f: \mathbb{R}^d \to \mathbb{R}$ at a finite point set $\Xi = \{x_1, x_2, \cdots, x_N\} \subset \mathbb{R}^d, d \geq 1$, is given. Scattered data interpolation entails finding an interpolant $s: \mathbb{R}^d \to \mathbb{R}$ satisfying

$$s(x_i) = f(x_i), \quad \text{for } i = 1,2, \cdots, N.$$

The radial kernel interpolation scheme works with a radial function $\varphi: \mathbb{R}_0^+ \to \mathbb{R}$, and the interpolant has the form

$$s(x) = \sum_{j=0}^{N} c_i \varphi(\varepsilon \| x - x_j \|)$$

where $\|\cdot\|$ is the Euclidean norm and ε is the shape parameter. This gives an $N \times N$ linear system

$$\sum_{j=0}^{N} c_j \varphi(\varepsilon \| x_i - x_j \|) = f(x_i), \ \text{for } i = 1,2, \cdots, N$$

which can be written in vector-matrix form as

$$A\boldsymbol{c} = \boldsymbol{f}$$

where $A = \varphi(\varepsilon\|x_i - x_j\|)$ is an $N \times N$ matrix and $\boldsymbol{c} = (c_1, c_2, \cdots, c_N)^T$. The matrix A is the interpolation matrix. We note that $\varphi(\varepsilon\|x_i - x_j\|) = \varphi(\varepsilon\|x_j - x_i\|)$ so that $A = A^T$. The interpolant is unique if and only if the matrix A is nonsingular. The existence of the interpolant has been shown in [9].

C. The Radial Kernel Collocation Method

In this section we build a collocation method for the solution of k^th order linear Fredholm integro-differential equations using radial kernels.

Here we assume that the solution of (1) can be expressed in the form of the radial kernel interpolant as

$$u(x) \approx \tilde{u}(x) = \sum_{i=0}^{m} c_i \varphi(\varepsilon\|x - x_i\|), \quad m > k \text{ and } x \in \mathbb{R} \tag{2}$$

We first of all select $m - k$ collocation points from the m data sites $x_1, \dots, x_m$. In our method, we let $\rho = \text{ceil}(k/2)$ and $q = \text{floor}(k/2)$, and so we use $x_\rho, \dots, x_{m-q}$ as the $m - k$ collocation points. The collocation method is given as

$$\sum_{n=0}^{k} P_n(x_j)\tilde{u}^{(n)}(x_j) = f(x_j) + \lambda \int_a^b k(x_j, t)\, \tilde{u}(x_j) dt, \quad j = \rho \dots m - q \tag{3}$$

We now substitute the approximate solution (2) into equation (3) and apply the collocation conditions to obtain

$$\sum_{n=0}^{k} P_n(x_j) \left[\sum_{i=0}^{m} c_i \varphi^{(n)}(\varepsilon\|x_j - x_i\|)\right] = f(x_j) + \lambda \int_a^b k(x_j, t) \left[\sum_{i=0}^{m} c_i \varphi(\varepsilon\|t - x_i\|)\right] dt, \; j = \rho \dots m - q$$

Re-arranging this yields

$$\sum_{i=0}^{m} c_i \left\{\sum_{n=0}^{k} P_n(x_j)\varphi^{(n)}(\varepsilon\|x_j - x_i\|) - \lambda \int_a^b k(x_j, t)\, \varphi(\varepsilon\|t - x_i\|) dt\right\} = f(x_j) \tag{4}$$

The integral in (4) is evaluated using a five-point Gauss-Legendre quadrature formula on the interval $[-1, 1]$ as

$$\int_{-1}^{1} g(x)\, dx = \sum_{l=1}^{5} w_l g(p_l)$$

where the $w_l's$ are the weights and the $p_l's$ are the integration points. To apply the rule over an arbitrary interval $[a, b]$, we use the change of variable [10]-[12]

$$t = \frac{a+b}{2} + \frac{a-b}{2} x \quad \text{where} \quad dt = \frac{a-b}{2} dx$$

Equation (4) can then be written as

$$\sum_{i=0}^{n} c_i \left\{\sum_{n=0}^{k} P_n(x_j)\varphi^{(n)}(\varepsilon\|x_j - x_i\|) - \frac{\lambda}{2}(a-b) \int_{-1}^{1} k\left(x_j, \frac{a+b}{2} + \frac{a-b}{2} x\right) \varphi\left(\varepsilon \left\|\frac{a+b}{2} + \frac{a-b}{2} x - x_i\right\|\right) dx\right\} = f(x_j) \tag{5}$$

and applying the quadrature rule we obtain

$$\sum_{i=0}^{n} c_i \left\{\sum_{n=0}^{k} P_n(x_j)\varphi^{(n)}(\varepsilon\|x_j - x_i\|) - \frac{\lambda}{2}(a-b) \left[\sum_{l=1}^{5} w_l k\left(x_j, \frac{a+b}{2} + \frac{a-b}{2} p_l\right) \varphi\left(\varepsilon \left\|\frac{a+b}{2} + \frac{a-b}{2} p_l - x_i\right\|\right)\right]\right\} = f(x_j) \tag{6}$$

The m coefficients $c_1, \dots, c_m$ of the approximate will require us solving a system of m linear equations.

Equation (6) gives $m-k$ linear equations in $c_1, \ldots, c_m$ while the remaining k equations are obtained by evaluating the approximate solution at the initial conditions. This gives

$$\tilde{u}^{(n)}(0) = \sum_{i=1}^{m} c_i \varphi^{(n)}(\varepsilon\|0 - x_i\|) = b_n, \quad n = 0, \ldots, k-1 \tag{7}$$

Equations (6) and (7) together yield the system of m equations in m unknowns [11].

D. Shape Parameter

In this work, we use the brute force method to compute a suitable estimate for the shape parameter ε. The brute force method consists of performing various interpolation experiments with different values of the shape parameter ε, and then choosing the value of the shape parameter that best minimizes the interpolation error. Thus, this is achieved by plotting the graph of interpolation error against the shape parameter. The minimum point on the curve gives the optimal value of the shape parameter ε [13].

III. Numerical Results

In this section wee demonstrate the viability of the radial kernel collocation method using two test problems.

Problem 1. Consider the second order integro-differential equation of Fredholm type with constant coefficients

$$u''(x) = e^x - x + \int_0^1 xy\, u(y)\, dy$$

with $u(0) = 1$, and $u'(0) = 1$ on the interval $[0,1]$. The exact solution is $u(x) = e^x$, for this problem $\rho = ceil\left(\frac{2}{2}\right) = 1$ and $q = floor\left(\frac{2}{2}\right) = 1$, $a = 0, b = 1$ and $k = 2$. We will solve the problem for $m = 11$. Thus, the collocation points will be $x_2, \ldots, x_{10}$. For $m = 11$ we have $h = \frac{1}{10}$. We used generalized Multiquadrics and linear Laguerre Gaussian kernels

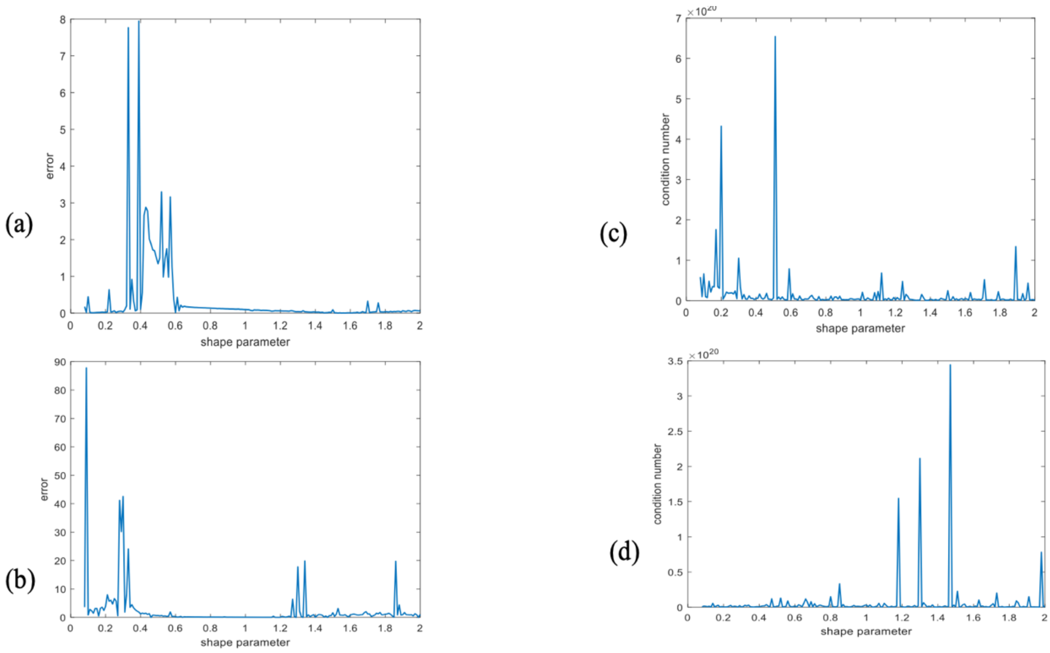

Fig. 1. (a) and (b) The Behaviour of the Shape Parameter with Respect to Maximum Error and (c) and (d) Condition Number, for the Collocation Solution of Problem 1 using Generalized Multiquadrics and Linear Laguerre-Gaussian Kernel with $m = 11, \varepsilon = 1.4$ Respectively.

TABLE I: Approximate Solution and Error for the Solution of Problem 1 Using Generalized Multiquadrics and Linear Laguerre Gaussian, $(M = 11, \varepsilon = 1.4)$

Points	x_i	Exact Solution	Approximate Solution (G. Multi-quadrics)	Absolute Error (G. Multi-quadrics)	Approximate Solution (LL Gaussian)	Absolute Error (LL Gaussian)
1	-1	0.368065	0.367879	1.851255×10^{-4}	0.367879	3.204464×10^{-4}
2	-4/5	0.449328	0.449329	1.052292×10^{-6}	0.449329	4.474686×10^{-7}
3	-3/5	0.548819	0.548812	7.418663×10^{-6}	0.548812	1.458191×10^{-5}
4	-2/5	0.670324	0.670320	3.794713×10^{-6}	0.670320	7.608273×10^{-6}
5	-1/5	0.818733	0.818731	2.253456×10^{-6}	0.818731	4.479410×10^{-6}
6	0	1.000000	1.000000	1.023000×10^{-12}	1.000000	9.535000×10^{-12}
7	1/5	1.221400	1.221403	2.463064×10^{-6}	1.221403	4.911218×10^{-6}
8	2/5	1.491821	1.491825	3.552700×10^{-6}	1.491824	7.122362×10^{-6}
9	3/5	1.822109	1.822119	9.342076×10^{-6}	1.822119	1.840068×10^{-6}
10	4/5	2.225553	2.225541	1.232835×10^{-5}	2.225541	2.216519×10^{-5}
11	1	2.717750	2.718282	5.313245×10^{-4}	2.718282	9.669150×10^{-4}

Problem 2. Consider the second order integro-differential equation of Fredholm type with variable coefficients

$$u''(x) + xu'(x) - xu(x) = e^x - 2\sin(x) + \int_{-1}^{1} \sin(x)\, e^{-y} u(y)\, dy$$

with $u(0) = 1$, and $u'(0) = 1$ on the interval [0,1]. The exact solution is $u(x) = e^x$, for this problem $k = 2$. $\alpha = \beta = 1$, $a = 1$, on the interval [0, 1]. The problem is solved for $m = 17$. Thus, the collocation points will be $x_2, \ldots, x_{16}$. We used generalized Multiquadrics and linear Laguerre Gaussian kernels.

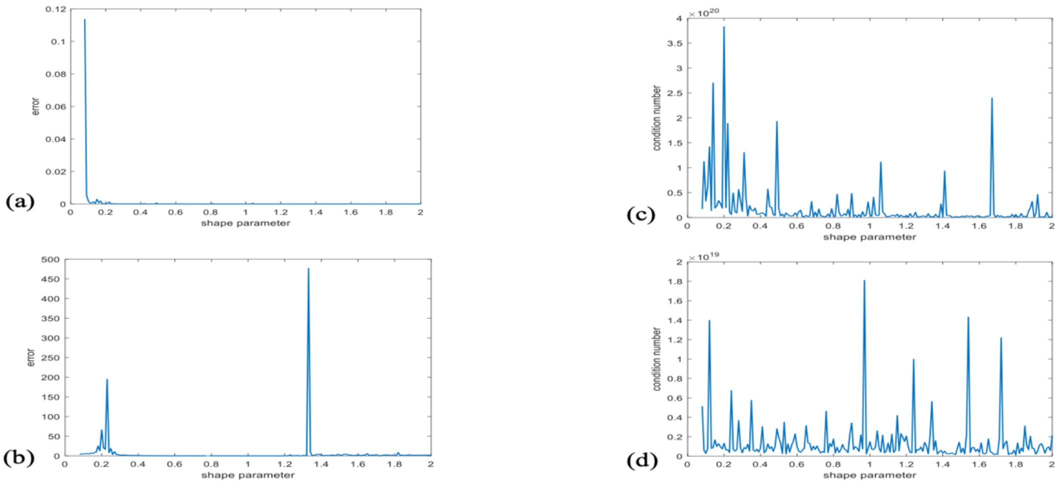

Fig. 2. (a) and (b) The Behaviour of the Shape Parameter with Respect to Error and (c) and (d) Condition Number for the Collocation Solution of Problem 2 using generalized Multiquadrics and Linear Laguerre Gaussian Kernel with $m = 17, 23$ respectively.

TABLE II: Approximate Solution and Error for the Solution of Fredholm Integro-Differential Equation Using Generalized Multiquadrics and Linear Laguerre Gaussian, $(m = 17, \varepsilon = 1)$

Points	x_i	Exact Solution	Approximate Solution (Multiquadrics)	Absolute Error (Multiquadrics)	Approximate Solution (LL Gaussian)	Absolute Error (LL Gaussian)
1	-1	0.367879	0.367879	8.968316×10^{-7}	0.367879	2.212018×10^{-6}
2	-7/8	0.416862	0.416862	2.7447000×10^{-8}	0.416862	5.183214×10^{-8}
3	-6/8	0.472367	0.472367	3.655369×10^{-7}	0.472367	8.563639×10^{-8}
4	-5/8	0.535261	0.535261	4.737887×10^{-8}	0.535261	8.148962×10^{-9}
5	-4/8	0.606531	0.606531	6.810450×10^{-8}	0.606531	1.080118×10^{-7}
6	-3/8	0.687289	0.687289	1.156636×10^{-7}	0.687289	4.004509×10^{-8}
7	-2/8	0.778801	0.778801	6.596934×10^{-8}	0.778801	7.288332×10^{-8}
8	-1/8	0.882497	0.882497	1.667250×10^{-7}	0.882497	5.581526×10^{-8}
9	0	1.000000	1.000000	1.096705×10^{-7}	1.000000	1.101389×10^{-7}
10	1/8	1.133148	1.133148	1.227157×10^{-7}	1.133148	1.491695×10^{-8}
11	2/8	1.284025	1.284025	1.402587×10^{-7}	1.284025	7.106465×10^{-7}
12	3/8	1.454991	1.454991	9.548872×10^{-8}	1.454991	1.270210×10^{-7}
13	4/8	1.648721	1.648721	1.933080×10^{-7}	1.648721	8.583456×10^{-8}
14	5/8	1.868246	1.868246	1.624123×10^{-7}	1.868246	1.817695×10^{-8}
15	6/8	2.117000	2.117000	2.188568×10^{-7}	2.117000	4.305514×10^{-8}
16	7/8	2.398875	2.398875	1.448256×10^{-7}	2.398875	1.317918×10^{-7}
17	1	2.718279	2.718282	2.285417×10^{-6}	2.718282	4.434408×10^{-6}

IV. DISCUSSION AND CONCLUSION

The results for the radial kernel collocation solution of Problem 1 using generalized Multiquadrics and linear Laguerre Gaussian m=11, ε=1.4 are provided in Table I. In Table I with m=11, ε=1.4, we observed that absolute errors for the two radial kernels are similar. We also observed same from the graph of the behaviour of shape parameter against error in Fig. 1 (a) and (b). The graph of the behaviour of shape parameter against condition number of the system matrix in Fig. 1 (c) and (d) showed that, both kernels generate a more stable system at their optimal shape parameter value ε=1.4. The error was more pronounced around the origin with small values of the shape parameter for both kernels, when the system was well-conditioned using LL Gaussian, while using generalized Multiquadrics kernel, the error was high between 0 and 0.4 in the same region where the system was also badly conditioned. The numerical results for problem 2 are also display in form of tables and graphs as shown in Table II and Fig. 2. In Table II with m=17,and ε=1, we observed that the absolute error for the two radial kernels are again similar. We also observed same from the graph of the behaviour of shape parameter against error in Fig. 2 (a) and (b). The graph of the behaviour of shape parameter against condition number of the system matrix in Fig. 2 (c) and (d) showed that, the system matrix at their optimal shape parameter value ε=1 is unstable. It can be concluded that this method of collocation presents a strong alternative technique for solving Fredholm integro-differential equations. It is also recommended that the optimal estimate for the shape parameter be obtain when using kernels that contain such parameter.

CONFLICT OF INTEREST

Authors declare that they do not have any conflict of interest.

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