Solving Differential Equations and Systems of Differential Equations with Inverse Laplace Transform

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***Abstract* — The inverse Laplace transform enables the solution of ordinary linear differential equations as well as systems of ordinary linear differentials with applications in the physical and engineering sciences. The Laplace transform is essentially an integral transform which is introduced with the help of a suitable generalized integral. The ultimate goal of this work is to introduce the reader to some of the basic ideas and applications for solving initially ordinary differential equations and then systems of ordinary linear differential equations.**

***Keywords* —Inverse Laplace transform; solving differential equations; solving systems of differential equations.**

I. INTRODUCTION

This paper deals with how to solve ordinary linear differential equations and ordinary linear differential equations by means of the Laplace inverse transform [1].

It is reported that in linear ordinary differential equations, the first derivative is usually denoted by $\dot{y}(t)$, the second $\ddot{y}(t)$, the third $\dddot{y}(t)$ and the $n-th$ $y^{(n)}(t)$ by and always $t \geq 0$.

II. SOLVING LINEAR DIFFERENTIAL EQUATIONS WITH INVERSE LAPLACE TRANSFORM

Let us be the linear differential equation

$$y^{(n)}(t)+a_1y^{(n-1)}(t)+a_2y^{(n-2)}(t)+...+a_{n-2}\ddot{y}(t)+a_{n-1}\dot{y}(t)+a_ny(t)=f(t)$$

where $a_1,a_2,...,a_{n-2},a_{n-1},a_n \in R$.

At least $y(t)$ be the solution to the above-mentioned linear differential equation which even verifies the initial conditions $y(0),\dot{y}(0),\ddot{y}(0),...,y^{(n-1)}(0)$.

Let the images of the functions $y(t)$ and $f(t)$ are $L\{y(s)\}$ and $L\{f(s)\}$, respectively. And for the initial derivatives functions $\dot{y}(t),\ddot{y}(t),...,y^{(n-1)}(t),y^{(n)}(t)$ the images are as follows:

$$sL\{y(t)\}-y(0)$$

$$s^2L\{y(t)\}-sy(0)-\dot{y}(0)$$

$$s^3L\{y(t)\}-s^2y(0)-s\dot{y}(0)-\ddot{y}(0)$$

....

$$s^nL\{y(t)\}-s^{n-1}y(0)-s^{n-2}\dot{y}(0)-s^{n-3}\ddot{y}(0)-...-sy^{(n-2)}(0)-y^{(n-1)}(0)$$

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Then the linear combination $y^{(n)}(t)+a_1y^{(n-1)}(t)+a_2y^{(n-2)}(t)+...+a_{n-2}\ddot{y}(t)+a_{n-1}\dot{y}(t)+a_ny(t)$ therefore:

$$\left[s^nL\{y(s)\}-s^{n-1}(0)-s^{n-2}\dot{y}(0)-s^{n-3}\ddot{y}(0)-...-s^2y^{(n-3)}(0)-sy^{(n-2)}(0)-y^{(n-1)}(0)\right]+$$
$$+a_1\left[s^{n-1}y(s)-s^{n-2}y(0)-s^{n-3}\dot{y}(0)-s^{n-4}\ddot{y}(0)-...-s^2y^{(n-4)}(0)-sy^{(n-3)}(0)-y^{(n-2)}(0)\right]+$$
$$+a_2\left[s^{n-2}y(s)-s^{n-3}y(0)-s^{n-4}\dot{y}(0)-...-s^2y^{(n-5)}(0)-sy^{(n-4)}(0)-y^{(n-3)}(0)\right]+$$
$$+...+$$
$$+a_{n-2}\left[s^2y(s)-sy(s)-\dot{y}(0)\right]+a_{n-1}\left[sy(s)-y(0)\right]+a_ny(s)$$

But $y^{(n)}(t)+a_1y^{(n-1)}(t)+a_2y^{(n-2)}(t)+...+a_{n-2}\ddot{y}(t)+a_{n-1}\dot{y}(t)+a_ny(t)=f(t)$, since the condition or function exists $y(t)$ is a solution of the linear differential equation.

So the image $L\{f(t)\}$ of function $f(t)$ will be as follows:

$$L\{f(t)\}=\left(s^n+a_1s^{n-1}+a_2s^{n-2}+...+a_{n-2}s^2+a_{n-1}s+a_n\right)y(s)-y(0)s^{n-1}-$$
$$-\left[\dot{y}(0)+a_1y(0)\right]s^{n-2}-\left[\ddot{y}(0)+a_1\dot{y}(0)+ay(0)\right]s^{n-3}-....-$$
$$-\left[y^{(n-2)}(0)+a_1y^{(n-3)}(0)+a_2y^{(n-4)}(0)+...+a_{n-2}y^{(n-3)}(0)\right]s-$$
$$-\left[y^{(n-1)}(0)+a_1y^{(n-2)}(0)+a_2y^{(n-3)}(0)+...+a_{n-2}\dot{y}(0)+a_{n-1}y(0)\right]$$

The last equation is the equation of the image of the linear equation. This equation can be written as follows: $\left(s^n+a_1s^{n-1}+a_2s^{n-2}+...+a_{n-2}s^2+a_{n-1}s+a_n\right)y(s)=f(t)+R_{n-1}(s)$, where $R_{n-1}(s)$ is a polynomial with variable s and its degree smaller $n-1$.

So we get the following: $y(t)=\dfrac{L\{f(t)\}+R_{n-1}(s)}{s^n+a_1s^{n-1}+a_2s^{n-2}+...+a_{n-2}s^2+a_{n-1}s+a_n}$

A special case of differential equation will follow, with integrals inside it. In particular, let the following differential equation: $\dot{y}(t)+ay(t)+b\int_0^t y(x)dx=f(t),\ a,b\in R$

Let $y(t)$ be the solution of the aforementioned differential equation and $L\{y(t)\}$ to be her image. The image of the first derivative $\dot{y}(t)$ will be the $sL\{y(t)\}-y(0)$, while the integral $\int_0^t y(x)dx$ will be the $\frac{1}{s}L\{y(t)\}$.

Then the image of the differential equation will be as follows $sL\{y(t)\}-y(0)+aL\{y(t)\}+b\frac{1}{s}L\{y(t)\}$.

But the linear combination $\dot{y}(t)+ay(t)+b\int_0^t y(x)dx=f(t)$ equals to $f(t)$, since there is a requirement $y(t)$ be the solution of the given differential equation. On the basis of all this implies that:

$$sL\{y(t)\}-y(0)+aL\{y(t)\}+b\frac{1}{s}L\{y(t)\}=L\{f(t)\}\Leftrightarrow$$
$$\Leftrightarrow s^2L\{y(t)\}-sy(0)+saL\{y(t)\}+bL\{y(t)\}=sL\{f(t)\}\Leftrightarrow$$
$$\Leftrightarrow L\{y(t)\}=\frac{sL\{f(t)\}+sy(0)}{s^2+sa+b}\Leftrightarrow y(t)=L^{-1}\left\{\frac{sL\{f(t)\}+sy(0)}{s^2+sa+b}\right\}$$

Finally, to facilitate the solution of differential equations by means of a Laplace transform, a related model: differential equation + initial conditions→linear algebraic equation→Laplace transform→solution to $L\{y(t)\}$→inverse Laplace transform→solution $y(t)$.

III. Solving Systems of Linear Differential equations with Inverse Laplace Transform

A. *Solving Systems of First Order Linear Differential Equations with Inverse Laplace Transform*

Let the following system of differential equations, if $x(0)=k,\ k\in R$ and $y(0)=l,\ l\in R$ [2]:

$$\begin{vmatrix}\dot{x}(t)=ax(t)+by(t)\\ \dot{y}(t)=cx(t)+dy(t)\end{vmatrix},\ \text{with } a,b,c,d\in R.$$

The solution is as follows:

$$\dot{x}(t)=ax(t)+by(t)\Leftrightarrow sL\{x(t)\}-x(0)=L\{ax(t)+by(t)\}\Leftrightarrow$$
$$\Leftrightarrow sL\{x(t)\}-k=L\{ax(t)\}+L\{by(t)\}\Leftrightarrow$$
$$\Leftrightarrow sL\{x(t)\}-k=aL\{x(t)\}+bL\{y(t)\}\Leftrightarrow$$
$$\Leftrightarrow (s-a)L\{x(t)\}=k+bL\{y(t)\}\Leftrightarrow L\{x(t)\}=\frac{k+bL\{y(t)\}}{s-a} \quad (1)$$

$$\dot{y}(t)=cx(t)+dy(t)\Leftrightarrow sL\{y(t)\}-y(0)=L\{cx(t)+dy(t)\}\Leftrightarrow$$
$$\Leftrightarrow sL\{y(t)\}-l=L\{ax(t)\}+L\{by(t)\}\Leftrightarrow$$
$$\Leftrightarrow sL\{y(t)\}-l=cL\{x(t)\}+dL\{y(t)\}\Leftrightarrow$$
$$\Leftrightarrow (s-d)\mathrm{L}\{y(t)\}=l+cL\{x(t)\}\Leftrightarrow \mathrm{L}\{y(t)\}=\frac{l+cL\{x(t)\}}{s-d} \quad (2)$$

Replace relationship (1) to (2) and have the following:

$$\mathrm{L}\{y(t)\}=\frac{l+c\dfrac{k+bL\{y(t)\}}{s-a}}{s-d}\Leftrightarrow \mathrm{L}\{y(t)\}=\frac{\dfrac{l(s-d)+ck+cbL\{y(t)\}}{s-a}}{s-d}\Leftrightarrow$$
$$\Leftrightarrow \mathrm{L}\{y(t)\}=\frac{l(s-d)+ck+cbL\{y(t)\}}{(s-a)(s-d)}\Leftrightarrow$$
$$\Leftrightarrow (s-a)(s-d)\mathrm{L}\{y(t)\}=l(s-d)+ck+cbL\{y(t)\}\Leftrightarrow$$
$$\Leftrightarrow \mathrm{L}\{y(t)\}=\frac{l(s-d)+ck}{(s-a)(s-d)-cb}\Leftrightarrow y(t)=\mathrm{L}^{-1}\left\{\frac{l(s-d)+ck}{(s-a)(s-d)-cb}\right\}$$

which is determined by the Laplace transform table. The value $x(t)$ is calculated in a way similar to relation (1).

B. *Solving Systems of Second Order Linear Differential Equations with Inverse Laplace Transform*

Let the following system of differential equations, if $x(0)=k$, $y(0)=l$, $\dot{x}(0)=m$ and $\dot{y}(0)=n$, with

$k,l,m,n \in R$ [3]: $\begin{vmatrix} \ddot{x}(t)+ax(t)+by(t)=0 \\ \ddot{y}(t)+cx(t)+dy(t)=0 \end{vmatrix}$, with $a,b,c,d \in R$.

We are asking for a solution $\ddot{x}(t)=s^2L\{x(t)\}-sx(0)-\dot{x}(0)$ and $\ddot{y}(t)=s^2L\{y(t)\}-sy(0)-\dot{y}(0)$ and then we have that:

$$\begin{vmatrix} \ddot{x}(t)+ax(t)+by(t)=0 \\ \ddot{y}(t)+cx(t)+dy(t)=0 \end{vmatrix} \Leftrightarrow \begin{vmatrix} s^2L\{x(t)\}-sx(0)-\dot{x}(0)+aL\{x(t)\}+bL\{y(t)\}=0 \\ s^2L\{y(t)\}-sy(0)-\dot{y}(0)+cL\{x(t)\}+dL\{y(t)\}=0 \end{vmatrix}$$

We then work as in systems of second order linear differential equations. If necessary, the inverse Laplace transformation is applied as appropriate.

IV. Practical Application of Differential Equations

Differential equations have great practical application in construction technology, the natural sciences, the economic sciences and many other disciplines. Two examples of applying the differential equations will be mentioned, one related to the spring-body system and the other to body buoyancy [4].

Example 1: Metal object mass m is hung by a fixed spring k. The object at the time t is exerting external force $F(t)$ downwards. The object also exerts on the spring a resistance force $F_s=-kx,\ k>0$, the air resistance force $F_a=-a\dot{x},\ a>0$ (a-proportionality constant) and the and the force of depth $B=mg$ (g-gravity acceleration). If at time t the spring-body system has an initial position $x_0=x(0)$ and an initial speed $v_0=\dot{x}(0)$ based on Newton's 2nd Law, the following differential equation is obtained: $m\ddot{x}=-kx-a\dot{x}+F(t)+mg$ or otherwise: $\ddot{x}+\frac{a}{m}\dot{x}+\frac{k}{m}x=F(t)+g,\ \dot{x}(0)=v_0,\ x(0)=x_0$.

Example 1: Examine a body of mass m immersed either partially or completely in a liquid with a density r. Such a body accepts two forces: a downward force due to gravity and an opposite force to lift, based on Archimedes' Principle: The body is balanced when the force of lifting the displaced fluid equals the force of gravity on the body.

For example, a cylindrical body with radius r, height H and body is submerged by h in height and balanced. In equilibrium, the volume of water displaced by the cylinder is $\pi r^2 h$, which provides a lifting force $\pi r^2 h\rho$ equal to the weight of the cylinder mg, i.e. $\pi r^2 h\rho = mg$ (ρ - density). If the cylinder is lifted out of the water by $x(t)$, then it is no longer in equilibrium. The downward or downward force in such a body remains mg but the upward or downward force decreases $\pi r^2\left[h-x(t)\right]\rho$. It follows from Newton's second law that: $m\ddot{x}(t)=\pi r^2\left[h-x(t)\right]\rho-mg \Leftrightarrow \ddot{x}(t)+\frac{\pi r^2\rho}{m}x(t)=0$.

V. Solved Exercises

Exercises 1: Solve the differential equation $\dot{y}(t)-2y(t)=4$, if $y(0)=0$.

Solution

$$\dot{y}(t)-2y(t)=4 \Leftrightarrow L\{\dot{y}(t)-2y(t)\}=L\{4\} \Leftrightarrow L\{\dot{y}(t)\}-L\{2y(t)\}=L\{4\} \Leftrightarrow$$

$$\Leftrightarrow sL\{y(t)\}-y(0)-2L\{y(t)\}=L\{4\} \Leftrightarrow L\{y(t)\}(s-2)=\frac{4}{s} \Leftrightarrow$$

$$\Leftrightarrow L\{y(t)\}=\frac{4}{s(s-2)} \Leftrightarrow L\{y(t)\}=4\frac{1}{s}\frac{1}{s-2} \Leftrightarrow y(t)=L^{-1}\left\{4\frac{1}{s}\frac{1}{s-2}\right\}$$

$$\frac{1}{s}\frac{1}{s-2}=\frac{A}{s}+\frac{B}{s-2}\Leftrightarrow 1=A(s-2)+Bs\Leftrightarrow 1=(A+B)s-2A\Leftrightarrow$$

$A+B=0$ and $-2A=1$ so $A=-\frac{1}{2}, B=\frac{1}{2}$

$$y(t)=4L^{-1}\left\{\frac{\frac{-1}{2}}{s}+\frac{\frac{1}{2}}{s-2}\right\}\Leftrightarrow y(t)=-2L^{-1}\left\{\frac{1}{s}\right\}+2L^{-1}\left\{\frac{1}{s-2}\right\}\Leftrightarrow$$

$$\Leftrightarrow y(t)=-2.1+2.e^{2t}\Leftrightarrow y(t)=-2+2.e^{2t}$$

Exercises 2: If $y(0)=0$ and $\dot{y}(0)=0$, solve the differential equation $\ddot{y}(t)+5\dot{y}(t)+4y(t)=t$, if $y(0)=0$ and $\dot{y}(0)=0$.

Solution

$$\ddot{y}(t)+5\dot{y}(t)+4y(t)=t\Leftrightarrow L\left\{\ddot{y}(t)+5\dot{y}(t)+4y(t)\right\}=L\{t\}\Leftrightarrow$$

$$\Leftrightarrow L\left\{\ddot{y}(t)\right\}+L\left\{5\dot{y}(t)\right\}+L\{4y(t)\}=L\{t\}\Leftrightarrow L\left\{\ddot{y}(t)\right\}+5L\left\{\dot{y}(t)\right\}+4L\{y(t)\}=L\{t\}\Leftrightarrow$$

$$\Leftrightarrow\left[s^2L\{y(t)\}-sy(0)-\dot{y}(0)\right]+5\left[sL\{y(t)\}-y(0)\right]+4L\{y(t)\}=L\{t\}\Leftrightarrow$$

$$\Leftrightarrow s^2L\{y(t)\}+5sL\{y(t)\}+4L\{y(t)\}=L\{t\}\Leftrightarrow$$

$$\Leftrightarrow s^2L\{y(t)\}+5sL\{y(t)\}+4L\{y(t)\}=L\{t\}\Leftrightarrow$$

$$\Leftrightarrow L\{y(t)\}(s^2+5s+4)=L\{t\}\Leftrightarrow L\{y(t)\}=\frac{1}{s^2+5s+4}L\{t\}\Leftrightarrow$$

$$\Leftrightarrow L\{y(t)\}=\frac{1}{s^2+5s+4}\frac{1}{s^2}\Leftrightarrow L\{y(t)\}=\frac{1}{(s+1)(s+4)}\frac{1}{s^2}\Leftrightarrow$$

$$\Leftrightarrow L\{y(t)\}=\frac{1}{s+1}\frac{1}{s+4}\frac{1}{s^2}\Leftrightarrow y(t)=L^{-1}\left\{\frac{1}{s+1}\frac{1}{s+4}\frac{1}{s^2}\right\}\Leftrightarrow$$

$$\Leftrightarrow s^2L\{y(t)\}+5sL\{y(t)\}+4L\{y(t)\}=L\{t\}\Leftrightarrow L\{y(t)\}(s^2+5s+4)=L\{t\}\Leftrightarrow$$

$$\Leftrightarrow L\{y(t)\}=\frac{1}{s^2+5s+4}L\{t\}\Leftrightarrow L\{y(t)\}=\frac{1}{s^2+5s+4}\frac{1}{s^2}\Leftrightarrow L\{y(t)\}=\frac{1}{(s+1)(s+4)}\frac{1}{s^2}\Leftrightarrow$$

$$\Leftrightarrow L\{y(t)\}=\frac{1}{s+1}\frac{1}{s+4}\frac{1}{s^2}\Leftrightarrow y(t)=L^{-1}\left\{\frac{1}{s+1}\frac{1}{s+4}\frac{1}{s^2}\right\}$$

So in terms of solving the given differential equation we will have the following:

$$y(t)=L^{-1}\left\{\frac{1}{s+1}\frac{1}{s+4}\frac{1}{s^2}\right\}\Leftrightarrow y(t)=L^{-1}\left\{\frac{\frac{1}{3}}{s+1}+\frac{-\frac{1}{48}}{s+4}+\frac{-\frac{5}{16}}{s}+\frac{\frac{1}{4}}{s^2}\right\}\Leftrightarrow$$

$$\Leftrightarrow y(t)=\frac{1}{3}L^{-1}\left\{\frac{1}{s+1}\right\}-\frac{1}{48}L^{-1}\left\{\frac{1}{s+4}\right\}-\frac{5}{16}L^{-1}\left\{\frac{1}{s}\right\}+\frac{1}{4}L^{-1}\left\{\frac{1}{s^2}\right\}\Leftrightarrow$$

$$\Leftrightarrow y(t)=\frac{1}{3}e^{-t}-\frac{1}{48}e^{-4t}-\frac{5}{16}+\frac{1}{4}t$$

Exercises 3: Solve the differential equation $y^{(n)}(t)-y(t)=0$, if $y(0)=1$ and $\dot{y}(0)=\ddot{y}(0)=\dddot{y}(0)=y^{(4)}(0)=0$.

Solution

$$L\{y(t)\}=s^5L\{y(t)\}-s^4 \Leftrightarrow s^5L\{y(t)\}-L\{y(t)\}=s^4 \Leftrightarrow L\{y(t)\}(s^5-1)=s^4 \Leftrightarrow$$
$$\Leftrightarrow L\{y(t)\}=\frac{s^4}{s^5-1} \Leftrightarrow L\{y(t)\}=\frac{1}{s}+\frac{1}{s^6}+\frac{1}{s^{11}}+\frac{1}{s^{21}}+\dots,\ |t|>1$$

From the $L\{f(t)\}=\sum_{n=0}^{+\infty}\frac{c_n}{s^{n+1}}$, for the function $y(t)$, which is the solution of the differential equation $y^{(n)}(t)-y(t)=0$, if $y(0)=1$ and $\dot{y}(0)=\ddot{y}(0)=\dddot{y}(0)=y^{(4)}(0)=0$, we will have that: $L\{y(t)\}=\frac{s^4}{s^5-1} \Leftrightarrow y(t)=L^{-1}\left\{\frac{s^4}{s^5-1}\right\} \Leftrightarrow y(t)=1+\frac{t^5}{5!}+\frac{t^{10}}{10!}+\frac{t^{20}}{20!}+\dots$.

Exercises 4: Solve the following system of differential equations, if $x(0)=1$ and $y(0)=1$:

$$\begin{cases}\dot{x}(t)=-2x(t)+2y(t)\\ \dot{y}(t)=-y(t)\end{cases}.$$

Solution

$$\dot{x}(t)=-2x(t)+2y(t) \Leftrightarrow L\{\dot{x}(t)\}=L\{-2x(t)+2y(t)\} \Leftrightarrow$$
$$\Leftrightarrow sL\{x(t)\}-x(0)=L\{-2x(t)+2y(t)\} \Leftrightarrow$$
$$\Leftrightarrow sL\{x(t)\}-1=-2L\{x(t)\}+2L\{y(t)\} \Leftrightarrow$$
$$\Leftrightarrow (s+2)L\{x(t)\}=2L\{y(t)\}+1 \Leftrightarrow L\{x(t)\}=\frac{2L\{y(t)\}+1}{s+2} \quad (1)$$
$$\dot{y}(t)=-y(t) \Leftrightarrow sL\{y(t)\}-y(0)=L\{-y(t)\} \Leftrightarrow sL\{y(t)\}-1=-L\{y(t)\} \Leftrightarrow$$
$$\Leftrightarrow sL\{y(t)\}+L\{y(t)\}=1 \Leftrightarrow L\{y(t)\}=\frac{1}{s+1} \quad (2)$$

From relationships (1), (2) we will have the following:

$$L\{x(t)\}=\frac{2L\{y(t)\}+1}{s+2} \Leftrightarrow L\{x(t)\}=\frac{2\frac{1}{s+1}+1}{s+2} \Leftrightarrow L\{x(t)\}=\frac{\frac{2+s+1}{s+1}}{s+2} \Leftrightarrow$$
$$\Leftrightarrow L\{x(t)\}=\frac{s+3}{(s+1)(s+2)} \Leftrightarrow L\{x(t)\}=\frac{s}{(s+1)(s+2)}+\frac{3}{(s+1)(s+2)} \Leftrightarrow$$
$$\Leftrightarrow x(t)=L^{-1}\left\{\frac{s}{(s+1)(s+2)}+\frac{3}{(s+1)(s+2)}\right\} \Leftrightarrow$$
$$\Leftrightarrow x(t)=L^{-1}\left\{\frac{s}{(s+1)(s+2)}\right\}+L^{-1}\left\{\frac{3}{(s+1)(s+2)}\right\} \Leftrightarrow$$
$$\Leftrightarrow x(t)=\frac{-2e^{-t}+e^{-2t}}{-1+2}+3\frac{e^{-t}-e^{-2t}}{-1+2} \Leftrightarrow x(t)=-2e^{-t}+e^{-2t}+3e^{-t}-3e^{-2t} \Leftrightarrow$$
$$\Leftrightarrow x(t)=e^{-t}-2e^{-2t}$$

From relation (2) we will have the following: $L\{y(t)\}=\frac{1}{s+1} \Leftrightarrow y(t)=L^{-1}\left\{\frac{1}{s+1}\right\} \Leftrightarrow y(t)=e^{-t}$.

Exercises 5: Solve the following system of differential equations, if $x(0)=4,\ \dot{x}(0)=0$ and $y(0)=2,\ \dot{y}(0)=0$: $\begin{vmatrix}\ddot{x}(t)+2x(t)-y(t)=0\\ \ddot{y}(t)+2y(t)-x(t)=0\end{vmatrix}$.

Solution

$$\begin{vmatrix}\ddot{x}(t)+2x(t)-y(t)=0\\ \ddot{y}(t)+2y(t)-x(t)=0\end{vmatrix} \Leftrightarrow \begin{vmatrix}s^2L\{x(t)\}-sx(0)-\dot{x}(0)+2L\{x(t)\}-L\{y(t)\}=0\\ s^2L\{y(t)\}-sy(0)-\dot{y}(0)+2L\{y(t)\}-L\{x(t)\}=0\end{vmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{vmatrix}s^2L\{x(t)\}-4s+2L\{x(t)\}-L\{y(t)\}=0\\ s^2L\{y(t)\}-2s+2L\{y(t)\}-L\{x(t)\}=0\end{vmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{vmatrix}(s^2+2)L\{x(t)\}-4s-L\{y(t)\}=0 \quad (1)\\ (s^2+2)L\{y(t)\}-2s-L\{x(t)\}=0 \quad (2)\end{vmatrix}$$

From relation (1) we will get the following:

$$(s^2+2)L\{x(t)\}-4s-L\{y(t)\}=0 \Leftrightarrow L\{y(t)\}=(s^2+2)L\{x(t)\}-4s \qquad (3)$$

From relation (2), by replacing (3), we will have the following:

$$(s^2+2)L\{y(t)\}-2s-L\{x(t)\}=0 \Leftrightarrow (s^2+2)\left[(s^2+2)L\{x(t)\}-4s\right]-2s-L\{x(t)\} \Leftrightarrow$$

$$\Leftrightarrow L\{x(t)\}=\frac{4s^3+10s}{(s^2+2)^2-1} \Leftrightarrow L\{x(t)\}=\frac{4s^3+10s}{(s^2+1)(s^2+3)} \Leftrightarrow x(t)=L^{-1}\left\{\frac{4s^3+10s}{(s^2+1)(s^2+3)}\right\}$$

For the calculation of the aforementioned results, in addition to the Laplace inverse transform method, the following procedure is suggested, taking into account that the triads s^2+1 and s^2+3 have complex roots.

$$\frac{4s^3+10s}{(s^2+1)(s^2+3)}=\frac{As+B}{s^2+1}+\frac{Cs+D}{s^2+3} \Leftrightarrow$$

$$\Leftrightarrow 4s^3+10s=(As+B)(s^2+3)+(Cs+D)(s^2+1) \Leftrightarrow$$

$$\Leftrightarrow 4s^3+10s=As^3+3As+Bs^2+3B+Cs^3+Cs+Ds^2+D \Leftrightarrow$$

$$\Leftrightarrow 4s^3+10s=(A+C)s^3+(B+D)s^2+(3A+C)s+(3B+D) \Leftrightarrow$$

$$\Leftrightarrow \begin{vmatrix}A+C=4\\ B+D=0\\ 3A+C=10\\ 3B+D=0\end{vmatrix} \Leftrightarrow \begin{vmatrix}A=3\\ B=0\\ C=1\\ D=0\end{vmatrix}$$

From relation (2) we get the following:

$$(s^2+2)L\{y(t)\}-2s-L\{x(t)\}=0 \Leftrightarrow L\{x(t)\}=(s^2+2)L\{y(t)\}-2s \quad (4)$$

From relation (1), by replacing (4), we will have the following:

$$(s^2+2)L\{x(t)\}-4s-L\{y(t)\}=0 \Leftrightarrow$$

$$\Leftrightarrow (s^2+2)\left[(s^2+2)L\{y(t)\}-2s\right]-4s-L\{y(t)\}=0 \Leftrightarrow$$

$$\Leftrightarrow L\{y(t)\}=\frac{2s^3+8s}{(s^2+2)^2-1} \Leftrightarrow L\{y(t)\}=\frac{2s^3+8s}{(s^2+1)(s^2+3)} \Leftrightarrow$$

$$\Leftrightarrow y(t)=L^{-1}\left\{\frac{2s^3+8s}{(s^2+1)(s^2+3)}\right\}$$

$$\frac{2s^3+8s}{(s^2+1)(s^2+3)}=\frac{As+B}{s^2+1}+\frac{Cs+D}{s^2+3}\Leftrightarrow 2s^3+8s=(As+B)(s^2+3)+(Cs+D)(s^2+1)\Leftrightarrow$$

$$\Leftrightarrow 2s^3+8s=As^3+3As+Bs^2+3B+Cs^3+Cs+Ds^2+D\Leftrightarrow$$

$$\Leftrightarrow 2s^3+8s=(A+C)s^3+(B+D)s^2+(3A+C)s+(3B+D)\Leftrightarrow$$

$$\Leftrightarrow \left|\begin{array}{l}A+C=2\\B+D=0\\3A+C=8\\3B+D=0\end{array}\right. \Leftrightarrow \left|\begin{array}{l}A=3\\B=0\\C=-1\\D=0\end{array}\right.$$

$$y(t)=L^{-1}\left\{\frac{2s^3+8s}{(s^2+1)(s^2+3)}\right\}\Leftrightarrow y(t)=L^{-1}\left\{\frac{3s}{s^2+1}+\frac{-s}{s^2+3}\right\}\Leftrightarrow$$

$$\Leftrightarrow y(t)=L^{-1}\left\{3\frac{s}{s^2+1^2}-\frac{s}{s^2+\left(\sqrt{3}\right)^2}\right\}\Leftrightarrow y(t)=3\cos t-\cos\sqrt{3}t$$

VI. Exercises Suggested for Solution Conclusion

Exercises 6: Solve the differential equation $\dot{y}(t)+2y(t)=10e^{3t}$, if $y(0)=1$.

Exercises 7: Solve the differential equation $\ddot{y}(t)-3\dot{y}(t)+2y(t)=2e^{3t}$, if $y(0)=0$ and $\dot{y}(0)=1$.

Exercises 8: Solve the differential equation $\ddot{y}(t)+2\dot{y}(t)+4y(t)=\sin 3t+e^{3t}$, if $y(0)=0$ and $\dot{y}(0)=-2$.

Exercises 9: Solve the following system of differential equations, if $x(0)=1$ and $y(0)=2$:

$$\left|\begin{array}{l}\dot{x}(t)=-x(t)+2y(t)\\\dot{y}(t)=-3y(t)\end{array}\right..$$

Exercises 10: Solve the following system of differential equations, if $x_0(0)=4,\ x_1(0)=0$ and $y_0(0)=2,\ y_1(0)=0$: $\left|\begin{array}{l}\ddot{x}(t)+2x(t)-y(t)=0\\\ddot{y}(t)+2y(t)-x(t)=0\end{array}\right..$

Conflict of Interest

Authors declare that they do not have any conflict of interest.

References

[1] Das AN. *Differential Equations with Introduction to Laplace Trans*form. India: New Central Book Agency; 2012.
[2] McLachlan NW. *Laplace Transforms and Their Applications to Differential Equations*. London: Dover; 2014.
[3] Veerarajan T. *Differential Equations and Laplace Transforms*. India: Yes Dee Publishing; 2020.
[4] Graf U. *Applied Laplace Transforms and z-Transforms for Scientists and Engineers: A Computational Approach using a Mathematica Package*. Switzerland: Birkhauser; 2004.