S-finite Coductor Rings

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Abstract

Recently, Anderson and Dumitrescu's S-finiteness has attracted the interest of several authors. Inspired by the work done by S. Glaz (See [8]), in this paper, we introduce and study a new concept called S-finite conductor rings, as an extension of the classical notion of finite conductor rings. Knowing that the latter is closely linked to the notion of weakly finite rings, so for compatibility reasons, it is convenient to define and study an extension of the concept of finite presentation, we will call it weakly S-finite conductor rings.

Index Terms

S-finitely generated, S-finitely presented, S-finite conductor rings, weakly S-finite conductor rings.

I. Introduction

THROUGHOUT this paper all rings are commutative with identity element.

Let R be a ring and S be a multiplicative subset of R, we denote $(a :_R b) = \{x \in R | \; xRb \subseteq Ra\}$ for all a and b in R and $Q(R)$ the total ring of quotients of R. According to [2], an R module M is called S-finite if there exists a finitely generated submodule N of M and $s \in S$, such that $sM \subseteq N$.

From [5], the interest of this notion stems from the fact that: if M be an S-finite. Then M_S is finite. Let M be an R-module is called S-finitely presented if there exist an exact sequence: $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$, where K is S-finite and F is a finitely generated free R-modules. Let I be an ideal of R, we note by $\mu(I)$ the minimal cardinal of a set of I generators. Let R be a ring, M be an R-module and $R \propto M$ be the set of pairs (r, m) with pairwise addition and multiplication given by: $(r, m)(r', m') = (rr', rm' + r'm)$. Henceforth we pose $R' = R \propto M$ called the trivial extension of R by M. R' is commutative ring with identity $(1, 0)$ (see for instance [1]).

The organization of this paper is follows: In section 2, we introduce and study notions "S-finite conductor rings" and "weakly S-finite conductor rings" and the relation between these two notions. In the case where R is a domain. We show that these notions are the same. An example is given where the ring is S-finite conductor not finite conductor. After we give several results equivalent to S-finite conductor ring. Section 3, we show in Theorem III and Corollary III, the notions "S-finite conductor rings" and "weakly S-finite conductor rings" pass from R' to R. On the other hand , the converse is not always true, as for example in Example III, but under certain conditions it will be possible. In Theorem III, we will follow another approach than that followed in [5, Proposition 2.7], to show this converse.

II. S-finite Conductor Rings

Let R be a ring and S a multiplicative subset of R. We say that R is an S-finite conductor ring if each 2-generated ideal is S-finitely presented. Let R be a ring and S a multiplicative subset of R. R is called weakly S-finite conductor ring if $Ra \cap Rb$ is S-finitely generated ideal of R.

1) Every finite conductor ring is an S-finite conductor ring.
2) Every weakly finite conductor ring is a weakly S-finite conductor ring.
3) Clearly R_S is finite conductor ring if R is S-finite conductor ring.
4) If R is an S-finite conductor ring, then R is a weakly S-finite conductor ring.
5) If R is an S-coherent ring, then R is an S-finite conductor ring.

Now, we give our main result, which characterize in many ways the notion of S-finite conductor rings. Let R be a ring. the following assertions are equivalent:

1) R is S-finite conductor ring.
2) $Ra \cap Rb$ and $(0 :_R c)$ are S-finitely generated ideals of R for all elements a, b and c of R.
3) Any (fractionary) ideal I of R with $\mu(I) \leq 2$ is S-finitely presented.
4) $(a :_R b)$ is S-finitely generated ideal of R for all elements a and b of $Q(R)$.

If R is a domain the above four properties are equivalent to:

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5) I^{-1} is S-finitely generated for any fractionary ideal I with $\mu(I) \leq 2$.

$(1 \Rightarrow 2)$ Let a, b and c tree elements of R. R is S-finite conductor ring. We consider the exact sequence: $0 \longrightarrow Ra \cap Rb \longrightarrow Ra \oplus Rb \longrightarrow Ra + Rb \longrightarrow 0$, we have by hypothesis $Ra + Rb$ is S-finitely generated. On the other hand $Ra \oplus Rb$ is finitely generated R-module. Then by [5, Theorem 2.5(3)] $Ra \cap Rb$ is an S-finitely generated. Moreover Rc is S-finitely presented. By the exact sequence $0 \longrightarrow (0 :_R c) \longrightarrow R \longrightarrow Rc \longrightarrow 0$, the ideal $(0 :_R c)$ is a S-finitely generated. $(2 \Rightarrow 3)$ Let $I = R\frac{a}{s} + R\frac{b}{t}$, where $a, b \in R$ and s, t are two regular elements of R. For all $x \in R$, Rx is S-finitely presented ideal of R (by the exact sequence: $0 \longrightarrow (0 :_R x) \longrightarrow R \longrightarrow Rx \longrightarrow 0$). On the other hand Ra and $R\frac{a}{s}$ are isomorphic. Then $R\frac{a}{s}$ is S-finitely presented, the same for $R\frac{b}{t}$. Thus $R\frac{a}{s} \oplus R\frac{b}{t}$ is S-finitely presented. On another hand $R\frac{a}{s} \cap R\frac{b}{t}$ is isomorphic to $Ra \cap Rb$ and $Ra \cap Rb$ is S-finitely generated. Then $R\frac{a}{s} \cap R\frac{b}{t}$ is S-finitely generated. Via the exact sequence: $0 \longrightarrow R\frac{a}{s} \cap R\frac{b}{t} \longrightarrow R\frac{a}{s} \oplus R\frac{b}{t} \longrightarrow I \longrightarrow 0$ and [5, Theorem 2.5(4)] I is S-finitely presented. $(3 \Rightarrow 4)$ Let x and y two elements of $Q(R)$. By the exact sequence: $0 \longrightarrow (x :_R y) \xrightarrow{f} Rx \oplus Ry \xrightarrow{g} Rx + Ry \longrightarrow 0$ (where for each $\lambda \in (x :_R y)$ there is $\mu \in R$ such that $\lambda y = \mu x$, so we pose $f(\lambda) = (\mu x, \lambda y)$ and $g(\alpha x, \beta y) = \alpha x - \beta y)$ and the [5, Theorem 2.5(5)] $(x :_R y)$ is S-finitely generated ideal of R (because $Rx \oplus Ry$ is finitely generated and $Rx + Ry$ is S-finitely presented). $(4 \Rightarrow 1)$ Let $I = Ra + Rb$ 2-generated ideal of R. By the exact sequence: $0 \longrightarrow (0 :_R a) \longrightarrow R \longrightarrow Ra \longrightarrow 0$, Ra is S-finitely presented. The same for Rb. Then the R-module $Ra \oplus Rb$ is S-finitely presented. By the exact sequence: $0 \longrightarrow (a :_R b) \longrightarrow Ra \oplus Rb \longrightarrow I \longrightarrow 0$, and [5, Theorem 2.5(4)] I is S-finitely presented. $(5 \Leftrightarrow 3)$ Let R be a domain and $I = Rx + Ry$ fractionary ideal with x, y no zero elements in $Q(R)$ $(\mu(I) \leq 2)$. Knowing that $I^{-1} = (R :_{Q(R)} I)$. It is easily shown that $I^{-1} = Rx^{-1} \cap Ry^{-1}$. We consider the exact sequence: $0 \longrightarrow I^{-1} \longrightarrow Rx^{-1} \oplus Ry^{-1} \longrightarrow Rx^{-1} + Ry^{-1} \longrightarrow 0$. The R-module $Rx^{-1} \oplus Ry^{-1}$ is finitely generated. [5, Theorem 2.5(4,5)] I^{-1} is S-finitely generated if only if $Rx^{-1} + Ry^{-1}$ is S-finitely presented. Which is equivalent to saying: every 2-generated fractionary ideal of R is S-finitely presented if only if its reverse is S-finitely generated. This shows the equivalence between (3) and (5).

The following result linking the notions S-finite conductor ring and weakly S-finite conductor ring. Let R be a ring. R is S-finite conductor ring if only if R is weakly S-finite conductor ring and each principal ideal of R is S-finitely presented. Immediately follows from 1 and 2 of Theorem II.

The following result the relation between "S-finite conductor rings" and "weakly S-finite conductor rings" when the ring is domain. Let R be a domain ring. R is S-finite conductor ring if only if R is weakly S-finite conductor ring. If R is a weakly S-finite conductor domain. Let $a \in R$, the ideal of R $(0 :_R a) = R$ if $a = 0$ or (0) if $a \neq 0$, then $(0 :_R a)$ is finitely generated ideal of R. By the exact sequence: $0 \longrightarrow (0 :_R a) \longrightarrow R \longrightarrow Ra \longrightarrow 0$, Ra is finitely presented. Let $I = Ra + Rb$ 2-generated ideal of R. We consider the exact sequence: $0 \longrightarrow Ra \cap Rb \longrightarrow Ra \oplus Rb \longrightarrow I \longrightarrow 0$ and R is weakly S-finite conductor ring. Then $Ra \cap Rb$ is S-finitely generated and the R-module $Ra \oplus Rb$ is finitely presented (because it's finite direct sum of finitely presented). By [5, Theorem 2.5(4)] I is S-finitely presented. This show that R is S-finite conductor ring. The converse is already demonstrate in the previous Theorem.

The following result shows that the notion of S-coherence on one ring induces the notion of S-finite conductor ring on another ring under certain conditions. Let R_1 and R_2 be rings and let $\varphi : R_1 \longrightarrow R_2$ be a ring homomorphism making R_2 as a finitely generated R_1-module and let S be a multiplicative subset of R_1. If R_1 is S-coherent ring, then R_2 is $\varphi(S)$-finite conductor ring.Let $I = R_2u + R_2v$ 2-generated ideal of R_2, where u and v two elements of R_2. R_2 is finitely generated as an R_1-module. Then I is finitely generated ideal of R_1 which is S-coherent ring. Then I is S-finitely presented as an R_1-module. By [5, Proposition 2.7] I is $\varphi(S)$- finitely presented as an R_2-module. This is show that R_2 is $\varphi(S)$-finite conductor ring.

We give two examples, the first gives the example of a ring S-finite conductor not finite conductor and the second gives the example of a ring is weakly S-finite conductor ring not finite conductor ring.

1) Let R_1 be an S_1-coherent ring, R_2 not finite conductor ring and $S = (S_1 \times \{0\}) \cup \{(1,1)\}$ who is multiplicative subset of the ring $R_1 \times R_2$. $R_1 \times R_2$ is S-finite conductor ring not finite conductor ring. otherwise if $R_1 \times R_2$ is finite conductor ring. So it's R_2, which is not the case.
2) With the same previous suggestions of (1) except that R_2 not weakly finite conductor ring . $R_1 \times R_2$ is weakly S-finite conductor ring not finite conductor ring. otherwise if $R_1 \times R_2$ is weakly finite conductor ring. So it's R_2, which is not the case.

In the third part we give an example of a weakly S-finite conductor ring not S-finite conductor ring.

III. Transfer of S-finite Conductor and Weakly S-finite Conductor Properties to the Trivial Extension

Throughout this part we note S' a multiplicative subset of $R' = R \propto M$ and $\pi_1 : R' \longrightarrow R$ the first projection which is a surjective morphism rings. Therefore $S = \pi_1(S')$ is a multiplicative subset of R.

1) If I be a finitely generated ideal of R' with $\mu(I) \leq n$ $(n \in^*)$. Then $\pi_1(I)$ is finitely generated ideal of R with $\mu(\pi_1(I)) \leq n$.
2) If I be an S'-finitely generated ideal of R'. Then $\pi_1(I)$ is S-finitely generated ideal of R.

The following two theorems show that the two notions S-finite conductor and weakly S-finite conductor pass from R' to R. If R' is S'-finite conductor ring. Then R is S-finite conductor ring.let $I = Ra + Rb$ 2-generated ideal of R. We consider $I' = R'(a, 0) + R'(b, 0)$, who is 2-generated ideal of R'. R' is S'-finite conductor ring, then I' is S'-finitely presented as an R'-module. on another side the morphism of rings π_1 making R a finitely generated as an R'-module (because $\pi_1(R') = R$). We show that: $0 \notin \pi_1(S') = S$, otherwise $0 \in \pi_1(S') = S$. Therefore it exists $(0, m) \in S'$ such that $\pi_1((0, m)) = 0 \in S$. on the other hand $(0, m)(0, m) = (0, 0) \in S'$. Which is absurd. then $0 \notin \pi_1(S') = S$. I' is also an R-module which is S'-finitely presented. By [5, Proposition 2.7] I' is S-finitely presented as an R-module, or $I' = I \times (aM + bM)$. Thus I is a direct summand of I'. Therefore I is S-finitely presented as an R-module. This is show that R is S-finite conductor. With the same demonstration, we can show the following property: If R' is finite conductor ring, then so is R. The following result can easily be deduced from Theorem III and Corollary II. If R' is weakly S'-finite conductor ring. Then R is weakly S-finite conductor ring. We given the converse of Theorem III in the case where M is an R-module S-coherent. In following Theorem we take S a multiplicative subset of R and $\varphi : R \longrightarrow R'$ a morphism of rings making R' a finitely generated as an R-module. Let $\varphi(S) = S'$, S' is a multiplicative subset of R'. Under the hypothesis previous. If R is S-finite conductor ring and $0 \notin S'$. Then R' is S'-finite conductor ring.let $I = R'(a, m) + R'(b, n)$ 2-generated ideal of R'. Then $I = (Ra+Rb)\times(Rm+Rn+aM+bN)$. Let $J = Ra + Rb$ who is 2-generated ideal of R. Then J is an S-finitely presented as an R-module (because R is S-finite conductor ring). On the other hand the morphism φ making R' a finitely generated as an R-module and $0 \notin S'$, by [5, Proposition 2.7] J is S'-finitely presented as an R'-module. Let $N = Rm + Rn + aM + bM$ which is a finitely generated submodule of M. Or M is S-coherent, then N is S-finitely presented as an R-module. For the same previous reason as that of J, N is S-finitely presented as an R'-module. Therefore I is S'-finitely presented as an R'-module. This show that R' is S'-finite conductor ring.

We give an example of weakly S-finite conductor ring not S-finite conductor ring. Let R be a weakly finite conductor ring not finite conductor ring and S subset of units of R (note that a S-finitely generated if only if it is finitely generated). Let M be an R-module, $R' = R \propto M$ the trivial extension of R by M. Let $S' = S \times \{0\}$ which is a multiplicative subset of R' (note that a S'-finitely generated if only if its image by π_1 is S-finitely generated if only if it is finitely generated). R is not finite conductor ring, then there exist $I = Ra + Rb$ 2-generated ideal of R not finitely presented. Clearly R' is weakly S'-finite conductor ring, because R is weakly finite conductor ring but not S'-finite conductor ring. By the absurd assume that R' is S'-finite conductor ring. R is not finite conductor ring. Then there exist $I = Ra + Rb$ 2-generated ideal of R not finitely presented. Let $I' = I \times \{0\} = R'(a, 0) + R'(b, 0)$ which is 2-generated as an R'-module, then I' is S'-finitely presented. Or π_1 is surjective morphism rings, thus π_1 making R finitely generated as an R'-module and $0 \notin \pi_1(S') = S$. Furthermore we can consider I as an R-module which is S'-finitely presented as an R'-module. By [5, Proposition 2.5(5)] I' is S-finitely presented as an R-module, then it is finitely presented as an R-module. Which is absurd. Consequently R' weakly S'-finite conductor ring not S'-finite conductor ring.

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