Compositions of Picture Fuzzy Relations with Application in Decision Making

Mohammad Kamrul Hasan, Abeda Sultana, and Nirmal Kanti Mitra

Abstract — **Picture fuzzy relation is an important and powerful concept which is suitable for describing correspondences between objects. It represents the strength of association of the elements of picture fuzzy sets. In this paper we have defined min-max composition for picture fuzzy relations and some properties are explored based on this definition. Also we have discussed some properties of max-min composition for picture fuzzy relations. Finally, an application is discussed as illustration to show how the picture fuzzy relation are applied in decision making.**

Keywords — **Composition of picture fuzzy relations; linguistic terms; picture fuzzy relation; picture fuzzy set.**

I. Introduction

In our daily life, we are to meet many things where most of them are vague than precise and those thinks always cannot be described by the concept of the conventional classical set theory. Most of the time this vagueness arises because of ambiguity, imprecision, partial information etc. To deal with this type of uncertainty, [1] introduced the concept of fuzzy set theory and then [2] developed the intuitionistic fuzzy set (IFS). However recently many researchers keep their concentration on picture fuzzy set which was developed by [3]. After the development of picture fuzzy set, it has been considered as a strong mathematical tool which is adequate in situations when human opinions involved more answers of the types yes, abstain, no and refusal. Therefore, the picture fuzzy set is the direct extension of fuzzy set and Intuitionistic fuzzy set by incorporating the concept of positive, negative and neutral membership degrees of an element. Fuzzy relation was initially introduced by [4] and then by [5]. Also it has been studied by a number of authors such as [6] and Zimmerman [7]. Then some scholars used it widely in many fields, such as decision making, fuzzy reasoning, fuzzy control, fuzzy diagnosis, clustering analysis [8]-[11] fuzzy comprehensive evaluation [12]-[14]. References [15] and [16] gave the definition of intuitionistic fuzzy relations and discussed some properties of them. In 2005, [17] further researched intuitionistic fuzzy relations and composition operation of intuitionistic fuzzy relations. [18] proposed the notion of picture fuzzy relations and studied their operations and properties and suggested distance measures between PFSs. Reference [19] investigated main fuzzy logic operators: negations, conjunctions, disjunctions and implications on picture fuzzy sets and also constructed main operations for fuzzy inference processes in picture fuzzy systems. Reference [20] properties of an involutive picture negator and some corresponding De Morgan fuzzy triples on picture fuzzy sets. Reference [21] investigate the classification of representable picture t-norms and picture t-conorms operators for picture fuzzy sets. Reference [22] discussed some aspects of equivalence picture fuzzy relation.

In this paper, min-max composition for picture fuzzy relations is defined and explored some properties on the basis of this definition. In addition, some properties of max-min composition for picture fuzzy relations are discussed thoroughly. A decision making problem is illustrated at the end of this paper.

II. Preliminaries

In this section, we recall some basic definitions which are used in later sections.

Definition 2.1[1] Let X be non-empty set. A fuzzy set A in X is given by

$$A = \{(x, \mu_A(x)) : x \in X\},$$

where $\mu_A : X \to [0, 1]$.

Definition 2.2 [2] An intuitionistic fuzzy set A in X is given by

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M. K. Hasan, Department of Mathematics, Jahangirnagar University, Savar, Bangladesh.
(corresponding e-mail: krul.habi@yahoo.com)
A. Sultana, Department of Mathematics, Jahangirnagar University, Savar, Bangladesh.
(e-mail: abedamathju@yahoo.co.uk).
N. K. Mitra, Department of Mathematics and Statistics, Bangladesh University of Business and Technology, Dhaka, Bangladesh.
(e-mail: dnkmitra@gmail.com)

$$A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\},$$

where $\mu_A: X \to [0, 1]$ and $\nu_A: X \to [0, 1]$, with the condition $0 \le \mu_A(x) + \nu_A(x) \le 1; \forall x \in X$.

The values $\mu_A(x)$ and $\nu_A(x)$ represent, respectively, the membership degree and non-membership degree of the element x to the set A.

For any intuitionistic fuzzy set A on the universal set X, for $x \in X$

$$\pi_A(x) = 1 - (\mu_A(x) + \nu_A(x))$$

is called the hesitancy degree (or intuitionistic fuzzy index) of an element x in A. It is the degree of indeterminacy membership of the element x whether belonging to A or not.

Obviously, $0 \le \pi_A(x) \le 1$ for any $x \in X$.

Particularly, $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is always valid for any fuzzy set A on the universal set X.

The set of all picture fuzzy sets in X will be denoted by $PFS(X)$.

Definition 2.3 [3] A picture fuzzy set A on a universe of discourse X is of the form

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) : x \in X\},$$

where $\mu_A(x) \in [0, 1]$ is called the degree of positive membership of x in A, $\eta_A(x) \in [0, 1]$ is called the degree of neutral membership of x in A and $\nu_A(x) \in [0, 1]$ is called the degree of negative membership of x in A, and where $\mu_A(x), \eta_A(x)$ and $\nu_A(x)$ satisfy the following condition:

$$0 \le \mu_A(x) + \eta_A(x) + \nu_A(x) \le 1; \forall x \in X.$$

Here $1 - (\mu_A(x) + \eta_A(x) + \nu_A(x)); \forall x \in X$ is called the degree of refusal membership of x in A.

The set of all picture fuzzy sets in X will be denoted by $PFS(X)$.

Definition 2.4 [3] Let $A, B \in PFS(X)$, then the subset, equality, the union, intersection and complement are defined as follows:

i. $A \subseteq B$ iff $\forall x \in X, \mu_A(x) \le \mu_B(x), \eta_A(x) \le \eta_B(x)$ and $\nu_A(x) \ge \nu_B(x)$;
ii. $A = B$ iff $\forall x \in X, \mu_A(x) = \mu_B(x), \eta_A(x) = \eta_B(x)$ and $\nu_A(x) = \nu_B(x)$;
iii. $A \cup B = \{(x, \max(\mu_A(x), \mu_B(x)), \min(\eta_A(x), \eta_B(x)), \min(\nu_A(x), \nu_B(x))) : x \in X\}$;
iv. $A \cap B = \{(x, \min(\mu_A(x), \mu_B(x)), \min(\eta_A(x), \eta_B(x)), \max(\nu_A(x), \nu_B(x))) : x \in X\}$;
v. $A^c = \{(x, \nu_A(x), \eta_A(x), \mu_A(x)) : x \in X\}$.

Definition 2.5 [3] Let X and Y be ordinary non-empty sets. A picture fuzzy relation R is a picture fuzzy subset of $X \times Y$ i,e R given by

$$R = \{((x, y), \mu_R(x, y), \eta_R(x, y), \nu_R(x, y)) : x \in X, y \in Y\}$$

where $\mu_R: X \times Y \longrightarrow [0,1]$, $\eta_R: X \times Y \longrightarrow [0,1]$, $\nu_R: X \times Y \longrightarrow [0,1]$ satisfying the condition

$0 \le \mu_R(x, y) + \eta_R(x, y) + \nu_R(x, y) \le 1$ for every $(x, y) \in (X \times Y)$.

Definition 2.6 [3] Let $R \in PFR(X \times Y)$. Then the inverse relation R^{-1}between Y and X:

$\mu_{R^{-1}}(y, x) = \mu_R(x, y)$, $\eta_{R^{-1}}(y, x) = \eta_R(x, y)$, $\nu_{R^{-1}}(y, x) = \nu_R(x, y)$, $\forall (x, y) \in X \times Y$.

Definition 2.7 [3] Let R and P be two picture fuzzy relations between X and Y, for every $(x, y) \in X \times Y$ we define:

(i) $R \le P \Leftrightarrow (\mu_R(x, y) \le \mu_P(x, y))$and $(\eta_R(x, y) \le \eta_P(x, y))$ and $(\nu_R(x, y) \ge \nu_P(x, y))$,
(ii) $R \vee P = \{(x, y), \mu_R(x, y) \vee \mu_P(x, y), \eta_R(x, y) \wedge \eta_P(x, y), \nu_R(x, y) \wedge \nu_P(x, y) : x \in X, y \in Y\}$,
(iii) $R \wedge P = \{(x, y), \mu_R(x, y) \wedge \mu_P(x, y), \eta_R(x, y) \wedge \eta_P(x, y), \nu_R(x, y) \vee \nu_P(x, y) : x \in X, y \in Y\}$,
(iv) $R^c = \{((x, y), \nu_R(x, y), \eta_R(x, y), \mu_R(x, y)) : x \in X, y \in Y\}$.

III. Compositions of Picture Fuzzy Relations

Definition 3.1 [3] Let $R \in PFR(X \times Y)$ and $S \in PFR(Y \times Z)$. Then the **max-min composition** of R and S is the picture fuzzy relation from X to Z defined as

$$R \circ S = \{((x,z), \mu_{R\circ S}(x,z), \eta_{R\circ S}(x,z), \nu_{R\circ S}(x,z)): x \in X, z \in Z\}, \text{ where}$$
$$\mu_{R\circ S}(x,z) = \vee_{y\in Y}\{\mu_S(x,y) \wedge \mu_R(y,z)\},$$
$$\eta_{R\circ S}(x,z) = \wedge_{y\in Y}\{\eta_S(x,y) \wedge \eta_R(y,z)\}$$
and
$$\nu_{R\circ S}(x,z) = \wedge_{y\in Y}\{\nu_S(x,y) \vee \nu_R(y,z)\}$$

whenever $0 \le \mu_{R\circ S}(x,z) + \eta_{R\circ S}(x,z) + \nu_{R\circ S}(x,z) \le 1$.

Definition 3.2 Let $R \in PFR(X \times Y)$ and $S \in PFR(Y \times Z)$, then the **min-max composition** of R and S is the picture fuzzy relation from X to Z defined as

$$R * S = \{(x,z), \mu_{R*S}(x,z), \eta_{R*S}(x,z), \nu_{R*S}(x,z): x \in X, z \in Z\}, \text{ where}$$
$$\mu_{R*S}(x,z) = \wedge_{y\in Y}\{\mu_S(x,y) \vee \mu_R(y,z)\}$$
$$\eta_{R*S}(x,z) = \wedge_{y\in Y}\{\eta_S(x,y) \wedge \eta_R(y,z)\}$$
And
$$\nu_{R*S}(x,z) = \vee_{y\in Y}\{\nu_S(x,y) \wedge \nu_R(y,z)\}$$

whenever $0 \le \mu_{R*S}(x,z) + \eta_{R*S}(x,z) + \nu_{R*S}(x,z) \le 1$.

Theorem 3.3 Let R, P be two elements of $PFR(X \times X)$, then $(R \circ P)^c = R^c * P^c$.

Proof: As $\mu_{R^c}(x,z) = \nu_R(x,z)$, $\eta_{R^c}(x,z) = \eta_R(x,z)$ and $\nu_{R^c}(x,z) = \mu_R(x,z)$; for every $(x,z) \in X \times X$.

We have,

$$R \circ P = \{(x,z), \vee_y\{\mu_P(x,y) \wedge \mu_R(y,z)\}, \wedge_y\{\eta_P(x,y) \wedge \eta_R(y,z)\}, \wedge_y\{\nu_P(x,y) \vee \nu_R(y,z)\}\}$$

Therefore,

$$(R \circ P)^c = \{(x,z), \wedge_y\{\nu_P(x,y) \vee \nu_R(y,z)\},\ \wedge_y\{\eta_P(x,y) \wedge \eta_R(y,z)\}, \vee_y\{\mu_P(x,y) \wedge \mu_R(y,z)\}\}$$
$$= R^c * P^c$$

Proposition 3.4 Let $R_1 \in PFR(X \times Y)$ and $R_2 \in PFR(Y \times Z)$, then $R_1 \circ R_2 \in PFR(X \times Z)$

Proof: For all $(x,z) \in X \times Z$, let proof

$$\mu_{R_1\circ R_2}(x,z) + \eta_{R_1\circ R_2}(x,z) + \nu_{R_1\circ R_2}(x,z) \le 1$$

For all $\epsilon > 0$, there exists $y^* \in Y$:

$$\mu_{R_1\circ R_2}(x,z) < \mu_{R_1}(x,y^*) \wedge \mu_{R_2}(y^*,z) + \epsilon$$

It is easily seen that

$$\eta_{R_1\circ R_2}(x,z) \le \eta_{R_1}(x,y^*) \wedge \eta_{R_2}(y^*,z)$$
and
$$\nu_{R_1\circ R_2}(x,z) \le \nu_{R_1}(x,y^*) \vee \nu_{R_2}(y^*,z).$$

Now,

$$\mu_{R_1\circ R_2}(x,z) + \eta_{R_1\circ R_2}(x,z) + \nu_{R_1\circ R_2}(x,z)$$

$$< \mu_{R_1}(x,y^*) \wedge \mu_{R_2}(y^*,z) + \eta_{R_1}(x,y^*) \wedge \eta_{R_2}(y^*,z) + \nu_{R_1}(x,y^*) \vee \nu_{R_2}(y^*,z) + \epsilon$$

Case 1: $\nu_{R_1}(x,y^*) \vee \nu_{R_2}(y^*,z) = \nu_{R_1}(x,y^*)$. Then

$$\mu_{R_1}(x,y^*) \wedge \mu_{R_2}(y^*,z) + \eta_{R_1}(x,y^*) \wedge \eta_{R_2}(y^*,z) + \nu_{R_1}(x,y^*) \vee \nu_{R_2}(y^*,z) + \epsilon$$

$$= \mu_{R_1}(x,y^*) \wedge \mu_{R_2}(y^*,z) + \eta_{R_1}(x,y^*) \wedge \eta_{R_2}(y^*,z) + \nu_{R_1}(x,y^*) + \epsilon$$
$$\le \mu_{R_1}(x,y^*) + \eta_{R_1}(x,y^*) + \nu_{R_1}(x,y^*) + \epsilon$$

$$\leq 1 + \epsilon.$$

Case 2: $\nu_{R_1}(x, y^*) \vee \nu_{R_2}(y^*, z) = \nu_{R_2}(y^*, z)$. Then

$$\mu_{R_1}(x, y^*) \wedge \mu_{R_2}(y^*, z) + \eta_{R_1}(x, y^*) \wedge \eta_{R_2}(y^*, z) + \nu_{R_1}(x, y^*) \vee \nu_{R_2}(y^*, z) + \epsilon$$
$$= \mu_{R_1}(x, y^*) \wedge \mu_{R_2}(y^*, z) + \eta_{R_1}(x, y^*) \wedge \eta_{R_2}(y^*, z) + \nu_{R_2}(y^*, z) + \epsilon$$
$$\leq \mu_{R_2}(y^*, z) + \eta_{R_2}(y^*, z) + \nu_{R_2}(y^*, z) + \epsilon$$
$$\leq 1 + \epsilon.$$

Then $\mu_{R_1 \circ R_2}(x, z) + \eta_{R_1 \circ R_2}(x, z) + \nu_{R_1 \circ R_2}(x, z) \leq 1 + \epsilon$ for all $\epsilon > o$.
Hence $\mu_{R_1 \circ R_2}(x, z) + \eta_{R_1 \circ R_2}(x, z) + \nu_{R_1 \circ R_2}(x, z) \leq 1$.

Theorem 3.5 For each $R \in PFR(X \times Y)$ and $S \in PFR(Y \times Z)$, $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$ is fulfilled.

Proof: $\mu_{(S \circ R)^{-1}}(z, x) = \mu_{(S \circ R)}(x, z)$

$= \vee_{y \in Y}\{\mu_R(x, y) \wedge \mu_S(y, z)\}$
$= \vee_{y \in Y}\{\mu_{R^{-1}}(y, x) \wedge \mu_{S^{-1}}(z, y)\}$
$= \vee_{y \in Y}\{\mu_{S^{-1}}(z, y) \wedge \mu_{R^{-1}}(y, x)\}$
$= \mu_{R^{-1} \circ S^{-1}}(z, x)$.
$\eta_{(S \circ R)^{-1}}(z, x) = \eta_{(S \circ R)}(x, z)$
$= \wedge_{y \in Y}\{\eta_R(x, y) \wedge \eta_S(y, z)\}$
$= \wedge_{y \in Y}\{\eta_{R^{-1}}(y, x) \wedge \eta_{S^{-1}}(z, y)\}$
$= \wedge_{y \in Y}\{\eta_{S^{-1}}(z, y) \wedge \eta_{R^{-1}}(y, x)\}$
$= \eta_{R^{-1} \circ S^{-1}}(z, x)$.
and
$\nu_{(S \circ R)^{-1}}(z, x) = \nu_{(S \circ R)}(x, z)$
$= \wedge_{y \in Y}\{\nu_R(x, y) \vee \nu_S(y, z)\}$
$= \wedge_{y \in Y}\{\nu_{R^{-1}}(y, x) \vee \nu_{S^{-1}}(z, y)\}$
$= \wedge_{y \in Y}\{\nu_{S^{-1}}(z, y) \vee \nu_{R^{-1}}(y, x)\}$
$= \nu_{R^{-1} \circ S^{-1}}(z, x)$.

Theorem 3.6 Let $P, Q, R \in PFR(X \times Y)$. Then

(i) $P \leq Q \Rightarrow P \circ R \leq Q \circ R$
(ii) $P \leq Q \Rightarrow R \circ P \leq R \circ Q$

Proof: (i) $\leq Q$, then
$\mu_P(y, z) \leq \mu_Q(y, z)$, $\eta_P(y, z) \leq \eta_Q(y, z)$ and $\nu_P(y, z) \geq \nu_Q(y, z)$
Now,
$\mu_{P \circ R}(x, z) = \vee_{y \in Y}\{\mu_R(x, y) \wedge \mu_P(y, z)\}$
$\leq \vee_{y \in Y}\{\mu_R(x, y) \wedge \mu_Q(y, z)\}$; as $\mu_P(y, z) \leq \mu_Q(y, z)$
$= \mu_{Q \circ R}(x, z)$
Similarly, we can prove that,

$\eta_{P \circ R}(x, z) \leq \eta_{Q \circ R}(x, z)$

Again,

$$\nu_{P \circ R}(x, z) = \wedge_{y \in Y}\{\nu_R(x, y) \vee \nu_P(y, z)\}$$
$$\geq \wedge_{y \in Y}\{\nu_R(x, y) \vee \nu_Q(y, z)\} \text{ ; as } \nu_P(y, z) \geq \nu_Q(y, z)$$
$$\geq \nu_{Q \circ R}(x, z)$$

The property (ii) can be proved in similar manner.

Theorem 3.7 Let $P, R \in PFR(Y \times Z)$, $Q \in PFR(X \times Y)$,then

(i) $(R \vee P) \circ Q = (R \circ Q) \vee (P \circ Q)$
(ii) $(R \wedge P) \circ Q = (R \circ Q) \wedge (P \circ Q)$

Proof: (i) In order to proof (i), we have to show that
(a) $\mu_{(R \vee P) \circ Q}(x, z) = \mu_{R \circ Q}(x, z) \vee \mu_{P \circ Q}(x, z)$

(b) $\eta_{(R\vee P)\circ Q}(x,z) = \eta_{R\circ Q}(x,z) \wedge \eta_{P\circ Q}(x,z)$
(c) $\nu_{(R\vee P)\circ Q}(x,z) = \nu_{R\circ Q}(x,z) \wedge \nu_{P\circ Q}(x,z)$.

(a) $\mu_{(R\vee P)\circ Q}(x,z) = \vee_{y\in Y}\{\mu_Q(x,y) \wedge (\mu_R(y,z) \vee \mu_P(y,z))\}$
$= \vee_{y\in Y}\left\{\left(\mu_Q(x,y) \wedge \mu_R(y,z)\right) \vee \left(\mu_Q(x,y) \wedge \mu_P(y,z)\right)\right\}$
$= \vee_{y\in Y}\left\{\left(\mu_Q(x,y) \wedge \mu_R(y,z)\right)\right\} \vee \vee_{y\in Y}\left\{\left(\mu_Q(x,y) \wedge \mu_P(y,z)\right)\right\}$
$= \mu_{R\circ Q}(x,z) \vee \mu_{P\circ Q}(x,z)$.

(b) $\eta_{(R\vee P)\circ Q}(x,z) = \wedge_{y\in Y}\{\eta_Q(x,y) \wedge (\eta_R(y,z) \wedge \eta_P(y,z))\}$
$= \wedge_{y\in Y}\left\{\left(\eta_Q(x,y) \wedge \eta_R(y,z)\right) \wedge \left(\eta_Q(x,y) \wedge \eta_P(y,z)\right)\right\}$
$= \wedge_{y\in Y}\left\{\left(\eta_Q(x,y) \wedge \eta_R(y,z)\right)\right\} \wedge \wedge_{y\in Y}\left\{\left(\eta_Q(x,y) \wedge \eta_P(y,z)\right)\right\}$
$= \eta_{R\circ Q}(x,z) \wedge \eta_{P\circ Q}(x,z)$.

(c) $\nu_{(R\vee P)\circ Q}(x,z) = \wedge_{y\in Y}\{\nu_Q(x,y) \vee (\nu_R(y,z) \wedge \nu_P(y,z))\}$
$= \wedge_{y\in Y}\left\{\left(\nu_Q(x,y) \vee \nu_R(y,z)\right) \wedge \left(\nu_Q(x,y) \vee \nu_P(y,z)\right)\right\}$
$= \wedge_{y\in Y}\left\{\left(\nu_Q(x,y) \vee \nu_R(y,z)\right)\right\} \wedge \wedge_{y\in Y}\left\{\left(\nu_Q(x,y) \vee \nu_P(y,z)\right)\right\}$
$= \nu_{R\circ Q}(x,z) \wedge \nu_{P\circ Q}(x,z); \forall (x,y) \in X \times Y$.

The property (ii) can be proved in similar manner.

Theorem 3.8 Let $Q \in PFR(X\times Y)$, $P \in PFR(Y\times Z)$, $R \in PFR(Z\times W)$, then $(R\circ P)\circ Q = R\circ (P\circ Q)$.

Proof: It is sufficient to prove that

1. $\mu_{(R\circ P)\circ Q}(x,z) = \mu_{R\circ (P\circ Q)}(x,z)$,
2. $\eta_{(R\circ P)\circ Q}(x,z) = \eta_{R\circ (P\circ Q)}(x,z)$,
3. $\nu_{(R\circ P)\circ Q}(x,z) = \nu_{R\circ (P\circ Q)}(x,z)$,

We have,

$\mu_{(R\circ P)\circ Q}(x,z) = \vee_{y\in Y}\{\mu_Q(x,y) \wedge \mu_{R\circ P}(y,z)\}$
$= \vee_y\left\{\mu_Q(x,y) \wedge \{\vee_t\{\mu_P(y,t) \wedge \mu_R(t,z)\}\}\right\}$
$= \vee_y\left\{\vee_t\left\{\mu_Q(x,y) \wedge \{\mu_P(y,t) \wedge \mu_R(t,z)\}\right\}\right\}$
$= \vee_t\{\vee_y\{[\mu_Q(x,y) \wedge \mu_P(y,t)] \wedge \mu_R(t,z)\}\}$
$= \vee_t\left\{\{\mu_{P\circ Q}(x,t) \wedge \mu_R(t,z)\}\right\}$
$= \mu_{R\circ (P\circ Q)}(x,z)\ ; \forall (x,z) \in X\times Z$.

$\eta_{(R\circ P)\circ Q}(x,z) = \wedge_{y\in Y}\{\eta_Q(x,y) \wedge \eta_{R\circ P}(y,z)\}$
$= \wedge_y\left\{\eta_Q(x,y) \wedge \{\wedge_t\{\eta_P(y,t) \wedge \eta_R(t,z)\}\}\right\}$
$= \wedge_y\left\{\wedge_t\left\{\eta_Q(x,y) \wedge \{\eta_P(y,t) \wedge \eta_R(t,z)\}\right\}\right\}$
$= \wedge_t\{\wedge_y\{[\eta_Q(x,y) \wedge \eta_P(y,t)] \wedge \eta_R(t,z)\}\}$
$= \wedge_t\{\eta_{P\circ Q}(x,t) \wedge \eta_R(t,z)\}$
$= \eta_{R\circ (P\circ Q)}(x,z)\ ; \forall (x,z) \in X\times Z$.

and $\nu_{(R\circ P)\circ Q}(x,z) = \wedge_{y\in Y}\{\nu_Q(x,y) \vee \nu_{R\circ P}(y,z)\}$
$= \wedge_y\left\{\nu_Q(x,y) \vee \{\wedge_t\{\nu_P(y,t) \vee \nu_R(t,z)\}\}\right\}$
$= \wedge_y\left\{\wedge_t\left\{\nu_Q(x,y) \vee \{\nu_P(y,t) \vee \nu_R(t,z)\}\right\}\right\}$
$= \wedge_t\{\wedge_y\{[\nu_Q(x,y) \vee \nu_P(y,t)] \vee \nu_R(t,z)\}\}$
$= \wedge_t\{\nu_{P\circ Q}(x,t) \vee \nu_R(t,z)\}$
$= \nu_{R\circ (P\circ Q)}(x,z)\ ; \forall (x,z) \in X\times Z$.

Theorem 3.9 For each $R \in PFR(X\times Y)$ and $S \in PFR(Y\times Z)$, $(S*R)^{-1} = R^{-1} * S^{-1}$ is fulfilled.

Proof: $\mu_{(S*R)^{-1}}(z,x) = \mu_{(S*R)}(x,z)$
$= \wedge_{y\in Y}\{\mu_R(x,y) \vee \mu_S(y,z)\}$
$= \wedge_{y\in Y}\{\mu_{R^{-1}}(y,x) \vee \mu_{S^{-1}}(z,y)\}$
$= \wedge_{y\in Y}\{\mu_{S^{-1}}(z,y) \vee \mu_{R^{-1}}(y,x)\}$
$= \mu_{R^{-1}*S^{-1}}(z,x)$.

$\eta_{(S*R)^{-1}}(z,x) = \eta_{(S*R)}(x,z)$

$= \wedge_{y \in Y}\{\eta_R(x,y) \wedge \eta_S(y,z)\}$
$= \wedge_{y \in Y}\{\eta_{R^{-1}}(y,x) \wedge \eta_{S^{-1}}(z,y)\}$
$= \wedge_{y \in Y}\{\eta_{S^{-1}}(z,y) \wedge \eta_{R^{-1}}(y,x)\}$
$= \eta_{R^{-1}*S^{-1}}(z,x).$

and $\nu_{(S*R)^{-1}}(z,x) = \nu_{(S*R)}(x,z)$
$= \vee_{y \in Y}\{\nu_R(x,y) \wedge \nu_S(y,z)\}$
$= \vee_{y \in Y}\{\nu_{R^{-1}}(y,x) \wedge \nu_{S^{-1}}(z,y)\}$
$= \vee_{y \in Y}\{\nu_{S^{-1}}(z,y) \wedge \nu_{R^{-1}}(y,x)\}$
$= \nu_{R^{-1}*S^{-1}}(z,x).$

Therefore, $(S*R)^{-1} = R^{-1} * S^{-1}$.

Theorem 3.10 Let $P, Q, R \in PFR(X \times Y)$. Then

(i) $P \le Q \Rightarrow P * R \le Q * R$
(ii) $P \le Q \Rightarrow R * P \le R * Q$

Proof: (i) $\le Q$, then

$\mu_P(y,z) \le \mu_Q(y,z)$, $\eta_P(y,z) \le \eta_Q(y,z)$ and $\nu_P(y,z) \ge \nu_Q(y,z)$

Now,

$\mu_{P*R}(x,z) = \wedge_{y \in Y}\{\mu_R(x,y) \vee \mu_P(y,z)\}$
$\le \wedge_{y \in Y}\{\mu_R(x,y) \vee \mu_Q(y,z)\}$; as $\mu_P(y,z) \le \mu_Q(y,z)$
$= \mu_{Q*R}(x,z)$

Similarly, we can prove that,

$\eta_{P*R}(x,z) \le \eta_{Q*R}(x,z)$

Again,

$\nu_{P*R}(x,z) = \vee_{y \in Y}\{\nu_R(x,y) \wedge \nu_P(y,z)\}$
$\ge \vee_{y \in Y}\{\nu_R(x,y) \wedge \nu_Q(y,z)\}$; as $\nu_P(y,z) \ge \nu_Q(y,z)$
$\ge \nu_{Q*R}(x,z)$

The property (ii) can be proved in similar manner.

Theorem 3.11 Let $P, R \in PFR(Y \times Z), Q \in PFR(X \times Y)$,then

(i) $(R \vee P) * Q = (R * Q) \vee (P * Q)$
(ii) $(R \wedge P) * Q = (R * Q) \wedge (P * Q)$

Proof: (i) In order to proof (i), we have to show that

(a) $\mu_{(R \vee P)*Q}(x,z) = \mu_{R*Q}(x,z) \vee \mu_{P*Q}(x,z)$
(b) $\eta_{(R \vee P)*Q}(x,z) = \eta_{R*Q}(x,z) \wedge \eta_{P*Q}(x,z)$
(c) $\nu_{(R \vee P)*Q}(x,z) = \nu_{R*Q}(x,z) \wedge \nu_{P*Q}(x,z)$.

(a) $\mu_{(R \vee P)*Q}(x,z) = \wedge_{y \in Y}\{\mu_Q(x,y) \vee (\mu_R(y,z) \vee \mu_P(y,z))\}$
$= \wedge_{y \in Y}\left\{\left(\mu_Q(x,y) \vee \mu_R(y,z)\right) \vee \left(\mu_Q(x,y) \vee \mu_P(y,z)\right)\right\}$
$= \wedge_{y \in Y}\left\{\left(\mu_Q(x,y) \vee \mu_R(y,z)\right)\right\} \vee \wedge_{y \in Y}\left\{\left(\mu_Q(x,y) \vee \mu_P(y,z)\right)\right\}$
$= \mu_{R*Q}(x,z) \vee \mu_{P*Q}(x,z).$

(b) $\eta_{(R \vee P)*Q}(x,z) = \wedge_{y \in Y}\{\eta_Q(x,y) \wedge (\eta_R(y,z) \wedge \eta_P(y,z))\}$
$= \wedge_{y \in Y}\left\{\left(\eta_Q(x,y) \wedge \eta_R(y,z)\right) \wedge \left(\eta_Q(x,y) \wedge \eta_P(y,z)\right)\right\}$
$= \wedge_{y \in Y}\left\{\left(\eta_Q(x,y) \wedge \eta_R(y,z)\right)\right\} \wedge \wedge_{y \in Y}\left\{\left(\eta_Q(x,y) \wedge \eta_P(y,z)\right)\right\}$
$= \eta_{R*Q}(x,z) \wedge \eta_{P*Q}(x,z).$

(c) $\nu_{(R \vee P)*Q}(x,z) = \vee_{y \in Y}\{\nu_Q(x,y) \wedge (\nu_R(y,z) \wedge \nu_P(y,z))\}$
$= \vee_{y \in Y}\left\{\left(\nu_Q(x,y) \wedge \nu_R(y,z)\right) \wedge \left(\nu_Q(x,y) \wedge \nu_P(y,z)\right)\right\}$
$= \vee_{y \in Y}\left\{\left(\nu_Q(x,y) \wedge \nu_R(y,z)\right)\right\} \wedge \vee_{y \in Y}\left\{\left(\nu_Q(x,y) \wedge \nu_P(y,z)\right)\right\}$

$$= \nu_{R*Q}(x,z) \wedge \nu_{P*Q}(x,z); \forall(x,y) \in X \times Y.$$

The property (ii) can be proved in similar manner.

Theorem 3.12 Let $Q \in PFR(X \times Y),\ P \in PFR(Y \times Z),\ R \in PFR(Z \times W)$, then

$$(R * P) * Q = R * (P * Q).$$

Proof: It is sufficient to prove that

1. $\mu_{(R*P)*Q}(x,z) = \mu_{R*(P*Q)}(x,z)$,
2. $\eta_{(R*P)*Q}(x,z) = \eta_{R*(P*Q)}(x,z)$,
3. $\nu_{(R*P)*Q}(x,z) = \nu_{R*(P*Q)}(x,z)$

We have,

$$\begin{aligned}
\mu_{(R*P)*Q}(x,z) &= \wedge_{y\in Y}\{\mu_Q(x,y) \vee \mu_{(R*P)}(y,z)\} \\
&= \wedge_{y\in Y}\left\{\mu_Q(x,y) \vee \{\wedge_t\{\mu_P(y,t) \vee \mu_R(t,z)\}\}\right\} \\
&= \wedge_{y\in Y}\{\wedge_t\{\mu_Q(x,y) \vee \{\mu_P(y,t) \vee \mu_R(t,z)\}\}\} \\
&= \wedge_t\{\wedge_y\{[\,\mu_Q(x,y) \vee \mu_P(y,t)] \vee \mu_R(t,z)\}\} \\
&= \wedge_t\left\{\{\,\mu_{P*Q}(x,t) \vee \mu_R(t,z)\,\}\right\} \\
&= \mu_{R*(P*Q)}(x,z)\ ; \forall(x,z) \in X \times Z.
\end{aligned}$$

$$\begin{aligned}
\eta_{(R*P)*Q}(x,z) &= \wedge_{y\in Y}\{\eta_Q(x,y) \wedge \eta_{R*P}(y,z)\} \\
&= \wedge_y\left\{\eta_Q(x,y) \wedge \{\wedge_t\{\eta_P(y,t) \wedge \eta_R(t,z)\}\}\right\} \\
&= \wedge_y\left\{\wedge_t\left\{\eta_Q(x,y) \wedge \{\eta_P(y,t) \wedge \eta_R(t,z)\}\right\}\right\} \\
&= \wedge_t\{\wedge_y\{[\eta_Q(x,y) \wedge \eta_P(y,t)] \wedge \eta_R(t,z)\}\} \\
&= \wedge_t\{\eta_{P\circ Q}(x,t) \wedge \eta_R(t,z)\} \\
&= \eta_{R*(P*Q)}(x,z)\ ; \forall(x,z) \in X \times Z.
\end{aligned}$$

and

$$\begin{aligned}
\nu_{(R*P)*Q}(x,z) &= \vee_{y\in Y}\{\nu_Q(x,y) \wedge \nu_{R*P}(y,z)\} \\
&= \vee_{y\in Y}\left\{\nu_Q(x,y) \wedge \{\vee_t\{\nu_P(y,t) \wedge \nu_R(t,z)\}\}\right\} \\
&= \vee_y\left\{\vee_t\left\{\nu_Q(x,y) \wedge \{\nu_P(y,t) \wedge \nu_R(t,z)\}\right\}\right\} \\
&= \vee_t\{\vee_y\{[\,\nu_Q(x,y) \wedge \nu_P(y,t)] \wedge \nu_R(t,z)\}\} \\
&= \vee_t\{\nu_{P*Q}(x,t) \wedge \nu_R(t,z)\} \\
&= \nu_{R*(P*Q)}(x,z)\ ; \forall(x,z) \in X \times Z.
\end{aligned}$$

IV. Application of the Picture Fuzzy Relations

A. *Methodology and Algorithm*

In selection of future higher study area, it cannot be represented in any numerical value. In this case vagueness arise, for example a student tells to the student counselor that his mathematical problem solving ability is "very high" with linguistic assessments. In this case any numerical value cannot represent this student's mathematical problem solving ability is "very high" but we can characterize student's mathematical problem solving ability by a picture fuzzy set as TABLE I.

TABLE I: Picture Fuzzy Linguistic Terms

Linguistic Terms	(μ,η,ν)
Extremely High (EH)	(0.9,0.0,0.1)
Very High (VH)	(0.8,0.1,0.1)
High (H)	(0.7,0.2,0.1)
Medium (M)	(0.5,0.2,0.3)
Low (L)	(0.4,0.2,0.4)
Very Low (VL)	(0.3,0.1,0.6)
Extremely Low (EL)	(0.1,0.1,0.8)

Hence we know that the membership function of picture fuzzy set can state vagueness information. The methodology involves mainly the following three jobs:

1. Determination of abilities.
2. Formulation of expert knowledge based on picture fuzzy relations.
3. Determination of higher study area selection on the basis of composition of picture fuzzy relations.

B. Procedure

Let $P = \{p_1, p_2, \cdots\cdots\cdots p_m\}$ be a set of m students for fixing their future higher study areas with a set of n subject knowledge skill $S = \{s_1, s_2, \cdots\cdots\cdots s_n\}$ and $D = \{d_1, d_2, \cdots\cdots\cdots d_k\}$ be a set of k study areas related to the subject knowledge skill. Now we build a picture fuzzy relation $R(P \times S)$ from P to S, where the entries are picture fuzzy sets $\tilde{a}_{ij} = (\mu_{ij}, \eta_{ij}, v_{ij})$ for $1 \le i \le m$ and $1 \le j \le n$. The student-subject knowledge skill relation $R(P \times S)$ is given as follows:

$$R(P \times S) = \begin{array}{c c} & \begin{array}{cccc} s_1 & s_2 & \cdots & s_n \end{array} \\ \begin{array}{c} p_1 \\ p_2 \\ \vdots \\ p_m \end{array} & \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} \\ \tilde{a}_{21} & \tilde{a}_{22} & \cdots & \tilde{a}_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{a}_{m1} & \tilde{a}_{m2} & \cdots & \tilde{a}_{mn} \end{bmatrix} \end{array}$$

Then, we create a relation $R_1(S \times D)$ from S to D which is called subject knowledge skill- study areas relation, where each element denote the weight of the subject knowledge skill for a certain study area. These elements are also taken as picture fuzzy set $\tilde{b}_{ij} = (\mu_{ij}, \eta_{ij}, v_{ij})$ for $1 \le i \le m$ and $1 \le j \le n$. The subject knowledge skill – study areas relation $R_1(S \times D)$ is given as follows:

$$R_1(S \times D) = \begin{array}{c c} & \begin{array}{cccc} d_1 & d_2 & \cdots & d_k \end{array} \\ \begin{array}{c} s_1 \\ s_2 \\ \vdots \\ s_n \end{array} & \begin{bmatrix} \tilde{b}_{11} & \tilde{b}_{12} & \cdots & \tilde{b}_{1k} \\ \tilde{b}_{21} & \tilde{b}_{22} & \cdots & \tilde{b}_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{b}_{n1} & \tilde{b}_{n2} & \cdots & \tilde{b}_{nk} \end{bmatrix} \end{array}$$

Now executing the transformation process $R \circ R_1$, we get the student-study area relation $R_2(P \times D)$ as follows:

$$R_2(P \times D) = R \circ R_1 = \begin{array}{c c} & \begin{array}{cccc} d_1 & d_2 & \cdots & d_k \end{array} \\ \begin{array}{c} p_1 \\ p_2 \\ \vdots \\ p_m \end{array} & \begin{bmatrix} \tilde{c}_{11} & \tilde{c}_{12} & \cdots & \tilde{c}_{1k} \\ \tilde{c}_{21} & \tilde{c}_{22} & \cdots & \tilde{c}_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{c}_{m1} & \tilde{c}_{m2} & \cdots & \tilde{c}_{mk} \end{bmatrix} \end{array}$$

Where $\tilde{c}_{ij} = \sum_{j=1}^{n} a_{ij} b_{jl}$ where $i = 1, 2, \cdots\cdots m$ and $l = 1, 2, \cdots\cdots k$. Then defuzzifying the matrix $R_3(P \times D)$ by (1), we get the classical student- study area relation as

$$R_3(P \times D) = \begin{array}{c c} & \begin{array}{cccc} d_1 & d_2 & \cdots & d_k \end{array} \\ \begin{array}{c} p_1 \\ p_2 \\ \vdots \\ p_m \end{array} & \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1k} \\ v_{21} & v_{22} & \cdots & v_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mk} \end{bmatrix} \end{array}$$

Finally if $\max_{1 \le l \le k} v_{il} = v_{is}$, then we terminate that the student p_i is selecting his study area d_s.

C. Algorithm

Step I: Input the picture fuzzy sets to get the student-subject knowledge skill relation $R(P \times S)$.

Step II: Input the picture fuzzy sets to get the subject knowledge skill - study areas relation $R_1(S \times D)$.

Step III: Execute the transformation operation $R_2(P \times D) = R(P \times S) \circ R_1(S \times D)$ to get the student-study area relation.

Step IV: Defuzzify all the elements of the matrix $R_2(P \times D)$ by (1) to get the matrix $R_3(P \times D)$.

Step V: Find s for which $v_{is} = \max_{l} v_{il}$.

Then we conclude that the student p_i is selecting his study area d_s. In case $\max_{1 \le l \le k} v_{il}$ happens for more than one value of l, then we can reassess the subjects knowledge to break the tie.

V. Case Study

In an institution, let there are four students $P = \{p_1, p_2, p_3, p_4\}$ wanted to determine their future higher study areas with respect to their subject knowledge skills: Mathematics, English Language, Biology, Physics, and Chemistry and the possible study areas regarding to the above subject knowledge skills be

Medicine, Pharmacy, Surgery and Anatomy. Let $S = \{s_1, s_2, s_3, s_4, s_5\}$ be a set of subject knowledge skills, where s_1, s_2, s_3, s_4 and s_5 represent: Mathematics, English Language, Biology, Physics and Chemistry, respectively. Let $D = \{d_1, d_2, d_3, d_4\}$ be a set of study areas, where d_1, d_2, d_3 and d_4 represent: Medicine, Pharmacy, Surgery and Anatomy, respectively. Observing the subject knowledge skills of four students', the student counselor can construct the following table with their linguistic assessments.

TABLE II: LINGUISTIC ASSESSMENTS OF FOUR STUDENTS

$R(P \times S)$	s_1: Mathematics	s_2: English Language	s_3: Biology	s_4: Physics	s_5: Chemistry
p_1	VL	H	EL	M	M
p_2	H	VH	L	M	EH
p_3	EH	M	M	H	VL
p_4	L	H	M	EH	M

Now, we have a corresponding picture fuzzy relation $R(P \times S)$ according to the rule of conversion between linguistic terms and numerical values showed in Table I. This relation is called student-subject knowledge skill relation and is given as follows:

$$R(P \times S) = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{array}{ccccc} s_1 & s_2 & s_3 & s_4 & s_5 \end{array} \\ \begin{bmatrix} (0.3,0.1,0.6) & (0.7,0.2,0.1) & (0.1,0.1,0.8) & (0.5,0.2,0.3) & (0.5,0.2,0.3) \\ (0.7,0.2,0.1) & (0.8,0.1,0.1) & (0.4,0.2,0.4) & (0.5,0.2,0.3) & (0.9,0.0,0.1) \\ (0.9,0.0,0.1) & (0.5,0.2,0.3) & (0.5,0.2,0.3) & (0.7,0.2,0.1) & (0.3,0.1,0.6) \\ (0.4,0.2,0.4) & (0.7,0.2,0.1) & (0.5,0.2,0.3) & (0.9,0.0,0.1) & (0.5,0.2,0.3) \end{bmatrix} \end{array}$$

Again, detecting the subject knowledge skill of four students', the career guidance officer can construct the following table with their linguistic assessments.

TABLE III: LINGUISTIC ASSESSMENTS FOR SUBJECT KNOWLEDGE SKILL TO STUDY AREAS

$R_1(S \times D)$	d_1: Medicine	d_2: Pharmacy	d_3: Surgery	d_4: Anatomy
s_1	EL	H	VH	VL
s_2	H	M	H	VH
s_3	M	H	EH	H
s_4	H	M	M	L
s_5	EH	EH	H	M

Now we make a picture fuzzy relation $R_1(S \times D)$. This relation is called subject knowledge skill - study areas relation and is given as follows:

$$R_1(S \times D) = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{array} \begin{array}{c} \begin{array}{cccc} d_1 & d_2 & d_3 & d_4 \end{array} \\ \begin{bmatrix} (0.1,0.1,0.8) & (0.7,0.2,0.1) & (0.8,0.1,0.1) & (0.3,0.1,0.6) \\ (0.7,0.2,0.1) & (0.5,0.2,0.3) & (0.7,0.2,0.1) & (0.8,0.1,0.1) \\ (0.5,0.2,0.3) & (0.7,0.2,0.1) & (0.9,0.0,0.1) & (0.7,0.2,0.1) \\ (0.7,0.2,0.1) & (0.5,0.2,0.3) & (0.5,0.2,0.3) & (0.4,0.2,0.4) \\ (0.9,0.0,0.1) & (0.9,0.0,0.1) & (0.7,0.2,0.1) & (0.5,0.2,0.3) \end{bmatrix} \end{array}$$

Then, executing the transformation operation $R_2(P \times D) = R(P \times S) \circ R_1(S \times D)$, we get the student-study areas matrix $R_2(P \times D)$ as follows:

$$R_2(P \times D) = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{array}{cccc} d_1 & d_2 & d_3 & d_4 \end{array} \\ \begin{bmatrix} (0.7, 0.0, 0.1) & (0.5, 0.0, 0.3) & (0.7, 0.0, 0.1) & (0.7, 0.1, 0.1) \\ (0.9, 0.0, 0.1) & (0.9, 0.0, 0.1) & (0.7, 0.0, 0.1) & (0.8, 0.0, 0.1) \\ (0.9, 0.0, 0.1) & (0.7, 0.0, 0.1) & (0.8, 0.0, 0.1) & (0.5, 0.0, 0.3) \\ (0.7, 0.0, 0.1) & (0.5, 0.0, 0.3) & (0.7, 0.0, 0.1) & (0.7, 0.0, 0.1) \end{bmatrix} \end{array}$$

Example for $R_2(p_1 \times d_1)$: $max\{min(0.3,0.1), min(0.7,0.7), min(0.1,0.5), min(0.5,0.7), min(0.5,0.9)\}$
$= max\{0.1,0.7,0.1,0.5,0.5\} = 0.7$
$min\{min(0.1,0.1), min(0.2,0.2), min(0.1,0.2), min(0.2,0.2), min(0.2,0.0)\}$
$= min\{0.1,0.2,0.1,0.2,0.0\} = 0.0$
$min\{max(0.6,0.8), max(0.1,0.1), max(0.8,0.3), max(0.3,0.1), max(0.3,0.1)\}$
$= min\{0.8,0.1,0.8,0.3,0.3\} = 0.1$

Now, defuzzifying the above matrix by (1), we get the classical student-study areas matrix as

$$R_3(P \times D) = \begin{array}{c c} & \begin{array}{cccc} d_1 & d_2 & d_3 & d_4 \end{array} \\ \begin{array}{c} p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} & \begin{bmatrix} 0.6 & 0.2 & 0.6 & 0.7 \\ 0.8 & 0.8 & 0.6 & 0.7 \\ 0.8 & 0.6 & 0.7 & 0.2 \\ 0.6 & 0.2 & 0.6 & 0.6 \end{bmatrix} \end{array}$$

From the above relation we conclude that p_1 can study either in Medicine or in Surgery, p_2 can study either in Medicine or in Pharmacy, p_3 can study only in Medicine and p_4 can study in any areas except Pharmacy. The results may be varied with the institutional rules and student counselor's assessments strategy in Table III.

VI. CONCLUSIONS

Picture fuzzy set is the latest generalization of both the fuzzy set and intuitionistic fuzzy set in the field of uncertainty to deal with ambiguity. The picture fuzzy relation plays a vital rule to make with association among the elements and decision making. In this paper, min-max composition for picture fuzzy relation is defined and explored some related properties using this definition. Moreover, some properties of max-min composition for picture fuzzy relations are discussed thoroughly. Finally, a decision making problem is discussed.

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CONFLICT OF INTEREST

Authors declare that they do not have any conflict of interest.

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