Stable Distribution of Multivariate Data

Vo Thi Truc Giang and Ho Dang Phuc

Abstract — The main theorem of the paper states that every stable random vector with marginal skewness parameters different from ±1 can be turned into a sub-Gaussian random vector by using an appropriately tailored transformation in multidimensional space. The theorem is used to create a formula on probability density function of stable random vector and to perform a procedure of testing the stable distribution of multivariate data. A dataset collected from the Nasdaq stock market is used to illustrate the proposed procedure.

Keywords — Data analysis; Heavy-tailed distribution; Portfolio selection; Stock market.

I. INTRODUCTION

Most of the traditional statistical analysis methods were developed under normality assumptions. The important class of normal distributions established itself in many years as a cornerstone of the most successful models in various areas of modern quantitative finance. These include, for instance, Markowitz’ pioneering model for portfolio selection and asset allocation [1], Ross’ arbitrage pricing theory [2], the famous capital asset pricing model of Sharpe [3], Treynor, Lintner and Mossin. In applications, however, normality is only a poor approximation of reality. Namely, whilst normal distributions are always symmetric around their mean, most of quantities usually concerned in empirical studies do not have symmetric distributions. For instance, observable returns mostly exhibit asymmetry in favor of large negative return deviations. Besides, normal distributions do not allow heavy tails, which are so common, especially in finance and risk management studies [4]–[9].

Stable distributions are asymmetric heavy-tailed extensions of normal distributions and have attracted a lot of attention in applied research [7], [9]–[13]. The univariate stable distributions are actually accessible by methods to estimate stable parameters and reliable programs to compute stable probability density functions, cumulative distribution functions, and quantiles for stable random variables [10], [14]–[16]. However, the use of the heavy-tailed models in practice has been restricted by the lack of tools for multivariate stable distributions.

The distribution function calculation problem of stable random vectors remains open in the general cases. Besides, in studies on portfolio selection and asset allocation, analysts must determine the joint probability density function and the cumulative distribution function of a linear combination of several stable random variables. It is worthy to notice that a random vector has stable distribution when and only when all linear combinations of its marginal have stable distribution with a common stable index. Thus, the problems of finding the multivariate stable probability density function and of testing the stable distribution of a random vector plays an important role in application. That convinces the aim of this paper to find a formula of multivariate stable probability density function estimation and to create a procedure of goodness-of-fit testing for a broader family of multivariate stable distributions.

Let \( X = (X_1, \ldots, X_d) \) be a random vector (rv hereafter) taking values in \( \mathbb{R}^d \), its cumulative distribution function (cdf) and probability density function (pdf) are denoted by \( F_X \) and \( f_X \). The coordinates \( X_1, \ldots, X_d \) are called marginal’s, simultaneously \( F_{X_1}, \ldots, F_{X_d} \) and \( f_{X_1}, \ldots, f_{X_d} \) are called marginal cdf’s and marginal pdf’s of \( X \). The rv \( X \) is said to have stable distribution if for every pair \( (X', X'') \) of independent rv’s identically distributed as \( X \), for every pair \( (a, b) \) of positive numbers, there exist a positive number \( c \) and a vector \( d \in \mathbb{R}^d \) such that \( cX' + bX'' \) has the same distribution as \( cX + d \). It is well known (Theorem 2.3.1 [17]) that the stable distribution of \( X \) is determined by a spectral measure \( \Lambda \) (a finite Borel measure on the unit sphere \( S_d \) in \( \mathbb{R}^d \)) and a shift vector \( \delta = (\delta_1, \ldots, \delta_d) \in \mathbb{R}^d \) through the representation

\[
\varphi_X(t) = \exp \left( -\int_{S_d} \psi_{\delta}(s) \Lambda(ds) + i\langle \delta, t \rangle \right)
\]  

where \( \varphi_X \) is the characteristic function of \( X \) defined by \( \varphi_X(t) = \mathbb{E} \exp(i \langle X, t \rangle) \), for \( t = (t_1, \ldots, t_d) \in \mathbb{R}^d \), \( \langle x, t \rangle = x_1 t_1 + \ldots + x_d t_d \), and

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\[ \psi_{\alpha}(u) = \begin{cases} |u|^\alpha \left( 1 - isign(u)\tan \frac{\pi \alpha}{2} \right) & \alpha \neq 1 \\ |u|^{\alpha} \left( 1 + i\frac{2}{\pi} \text{sign}(u) \ln |u| \right) & \alpha = 1. \end{cases} \]

We denote \( X \sim \mathcal{S}(\alpha; \Lambda; \mathbf{b}) \) to mark (1) is valid. Moreover, \( X \) is said to be \( \alpha \)-stable. Especially, characteristic function of \( \alpha \)-stable random variable has the form

\[ \varphi_{\alpha}(u) = \E \exp(\i uX) = \begin{cases} \exp \left( -\gamma^\alpha |u|^\alpha \left( 1 - i\beta \left( \frac{\tan \frac{\pi \alpha}{2}}{\alpha} \right) \text{sign}(u) + i\delta u \right) \right) & \alpha \neq 1 \\ \exp \left( -\gamma |u| \left( 1 + i\beta \frac{2}{\pi} \text{sign}(u) \ln |u| + i\delta u \right) \right) & \alpha = 1. \end{cases} \]

with fixed \( \beta \in [-1; 1], \gamma > 0 \) and \( \delta \in \mathbb{R} \). Then the distribution of \( X \) is uniquely determined by the parameters \( \alpha, \beta, \gamma \), \( \delta \), we write \( X \sim \mathcal{S}(\alpha; \beta; \gamma; \delta) \). Usually, \( \alpha \) is called the stable index, meanwhile \( \beta, \gamma \) and \( \delta \) are named as the skewness, the scale and the location parameters of \( X \), respectively.

## II. STABLE RANDOM VECTOR AND SUB-GAUSSIAN RANDOM VECTOR

For fixed \( \alpha \in (0, 2] \), let \( A \sim \mathcal{S}(\alpha/2; 1; \left[ \cos \left( \pi \alpha/4 \right) \right]^2; 0) \) be a positive \( \alpha/2 \)-stable random variable and \( G = (G_1, \ldots, G_d) \) be a zero-mean Gaussian vector independent of \( A \). Then \( X = (A^{1/2}G_1, \ldots, A^{1/2}G_d) \) is called a sub-Gaussian rv. By virtue of Theorems 1.3.1 and 2.1.5 [17], \( X \) is an \( \alpha \)-stable rv. Moreover, the following result is an immediate consequence of Proposition 2.5.5 [17].

**Lemma 2.1.** For \( \alpha \in (0, 2] \), let \( X \sim \mathcal{S}(\alpha; \Lambda; \mathbf{b}) \) be an \( \alpha \)-stable rv in \( \mathbb{R}^d \). Suppose that the distribution of \( X \) is isotropic, which means \( X \overset{d}{=} M\mathbf{X} \) for all orthonormal \( d \times d \) matrices \( M \) (corresponding to linear rotations around the origin \( \mathbf{0} \)), where \( \overset{d}{=} \) denotes the equality in distribution. Then \( X \) is sub-Gaussian with Gaussian vector \( \mathbf{G} \) having iid components \( G_i, i = 1, \ldots, d \).

In the sequel, we use repeatedly the couple of the polar representation mappings \( B = (B_0, \ldots, B_{d-1}) \) and \( D = (D_1, \ldots, D_d) \) defined by (2) and (3) as the follows. For \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \setminus \{0\} \), let

\[
\begin{align*}
\left\{ \begin{array}{l}
\mathbf{r} = B_0(x) = |x| = \sqrt{x_1^2 + \cdots + x_d^2}, \\
\theta_i = B_i(x) = \arccot \left( \frac{x_i}{\sqrt{x_{i+1}^2 + \cdots + x_d^2}} \right), i = 1, 2, \ldots, d - 2, \\
\theta_{d-1} = B_{d-1}(x) = 2\arccot \left( \frac{x_{d-1}}{x_d} \right)
\end{array} \right.
\end{align*}
\]

Then \( B = (B_0, \ldots, B_{d-1}) : \mathbb{R}^d \setminus \{0\} \to \mathbb{S}^{d-1} \times \{0\} \) is a bijective mapping.

\[
\mathbb{S}^{d-1} = \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d : x_1^2 + \cdots + x_d^2 = 1, x_d > 0 \right\}
\]

Inversely, each point \( x \in \mathbb{R}^d \setminus \{0\} \) is of the form

\[
\begin{align*}
\left\{ \begin{array}{l}
x_1 = D_1(r, \theta_1, \ldots, \theta_{d-1}) = r \cos \theta_1, \\
x_2 = D_2(r, \theta_1, \ldots, \theta_{d-1}) = r \cos \theta_1 \cos \theta_2, \\
\vdots \\
x_{d-1} = D_{d-1}(r, \theta_1, \ldots, \theta_{d-1}) = r \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{d-2} \cos \theta_{d-1}, \\
x_d = D_d(r, \theta_1, \ldots, \theta_{d-1}) = r \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{d-2} \sin \theta_{d-1},
\end{array} \right.
\end{align*}
\]

with \( 0 \leq \theta_1 < \pi, \ldots, 0 \leq \theta_d < \pi, 0 \leq \theta_{d-1} < 2\pi \). Then the mapping \( D = (D_1, \ldots, D_d) : \mathbb{S}^{d-1} \times \{0\} \to \mathbb{R}^d \) is the inverse transformation of \( B \).

**Lemma 2.2.** Given \( \alpha \in (0, 2] \) and a symmetric \( \alpha \)-stable rv \( Z = (Z_1, \ldots, Z_d) \), let \( B(Z) = (R, \Theta_1, \ldots, \Theta_{d-1}) \). Suppose that there exists an invertible differentiable transformation \( Q : \mathbb{S}^{d-1} \to \mathbb{S}^{d-1} \) such that \( Y = D(R, Q(\Theta_1, \ldots, \Theta_{d-1})) \) is symmetric. Then \( Y \) has an \( \alpha \)-stable distribution.

**Proof.** From the symmetry of \( Z \) and \( Y \) we see their characteristic function \( \varphi_Z \) and \( \varphi_Y \) taking only real values and

\[ \varphi_Z(t) = \E \cos(\langle Z, t \rangle), \quad \varphi_Y(t) = \E \cos(\langle Y, t \rangle) \]

for \( t = (t_1, \ldots, t_d) \in \mathbb{R}^d \), where \( \langle Z, t \rangle = z_1t_1 + \cdots + z_dt_d \) for \( Z = (z_1, \ldots, z_d) \). Meantime, the spectral representation of symmetric stable multivariate distributions (see e.g. Theorem 2.4.3 [17]) confirms that

\[ \varphi_Z(t) = \exp \left( -\int_{\mathbb{S}^d} \|s, t\|^\alpha \Lambda(ds) \right) \]

with a spectral measure \( \Lambda \) on \( \mathbb{S}^d \). Looking at the polar representations.
where \(0 \leq \theta_1 < \pi, \ldots, 0 \leq \theta_{d-2} < \pi, 0 \leq \theta_{d-1} < 2\pi; 0 \leq \eta_1 < \pi, \ldots, 0 \leq \eta_{d-2} < \pi, 0 \leq \eta_{d-1} < 2\pi\) and \(r = ||z||; u = ||t||\), we see

\[
(z, t) = ru \cdot \nu(\theta, \eta),
\]

the function \(\nu\) depends only of the arguments \(\Theta = (\theta_1, \ldots, \theta_{d-1}), \eta = (\eta_1, \ldots, \eta_{d-1})\) from \(\mathbb{S}^{d-1}\). The existence of pdf \(f_Z\) of the stable rv \(Z\) and the assumption that \(Q\) is differentiable imply the pdf \(f_Y\) of the rv \(Y\) exists, that yields

\[
f'_x(r, \theta_1, \ldots, \theta_{d-1}) = f'_x(r, Q(\Theta))J_Q,
\]

where \(f^*\) means the polar coordinates form of a multivariate function \(f\) and \(J_Q\) denotes the Jacobian of \(Q\), that is dependent only on the arguments \(\theta_1, \ldots, \theta_{d-1}\). Therefore (4) can be rewritten as

\[
\phi_Z(t) = \int_0^\infty \int_0^\infty \cos(\nu \cdot \nu(\theta, \eta)) f'_x(r, \theta)drd\theta
\]

and

\[
\phi_X(t) = \exp\left(-\int_0^\infty \int_0^\infty \cos(\nu \cdot \nu(Q(\theta), \eta)) f'_x(r, Q(\theta))J_Qdrd\theta \right)
\]

Simultaneously, (5) implies

\[
\phi_X(t) = \exp\left(-\int_0^\infty \int_0^\infty \cos(\nu \cdot \nu(\Theta, \eta)) A'(d\Theta) \right) = \exp\left(-\int_0^\infty \int_0^\infty \cos(s, t) A'(d\Theta) \right)
\]

for some spectral measure \(A'\) defined on \(\mathbb{S}_d\). Consequently, the rv \(Y\) is \(\alpha\)-stable by virtue of Theorem 2.4.3. [17].

For a given cdf \(G: \mathbb{R} \rightarrow [0, 1]\), let \(G^-(y) = \inf\{x: G(x) \geq y\}\) be its generalized inverse. The copula of rv \(X\), denoted by \(C_X\), can be defined by

\[
C_X(t_1, \ldots, t_d) = F_X\left(F^{-1}_{X_1}(t_1), \ldots, F^{-1}_{X_d}(t_d)\right),
\]

for \(0 \leq t_1, \ldots, t_d \leq 1\). Then we have (see also Sklar’s Theorem [18])

\[
F_X(x_1, \ldots, x_d) = C_X\left(F_{X_1}(x_1), \ldots, F_{X_d}(x_d)\right),
\]

for \(x_1, \ldots, x_d \in \mathbb{R} = [-\infty; +\infty]\). Moreover, if \(X\) is continuous then \(F^{-1}_x = F^{-1}_{X_k}\) for \(k = 1, \ldots, d\). By virtue of (9), if \(C\) is a copula of any stable rv and \(X_1, \ldots, X_d\) are stable then

\[
F_X(x_1, \ldots, x_d) = C\left(F_{X_1}(x_1), \ldots, F_{X_d}(x_d)\right)
\]

is a cdf of a stable rv.

THEOREM 2.3. Let \(X\) be a rv with stable distribution such that its marginals have skewness parameters different from \(\pm 1\). Then there exists for \(X\) an invertible differentiable transformation \(K: \mathbb{R}^d \rightarrow \mathbb{R}^d\) such that the rv \(Y = K(X)\) is a sub-Gaussian rv.

PROOF. Let \(X_1, \ldots, X_d\) be a stable rv of marginals \(X_k \sim S(\alpha; \beta_k; \gamma_k; \delta_k)\), \(0 < \alpha < 2; -1 < \beta_k < 1; 0 < \gamma_k < \infty, k = 1, \ldots, d\). Let \(X_0 \sim S(\alpha; 0; 0; 1)\) denote a standard stable random variable. From the well-known fact (Property 1.2.14 [17]) that if a stable random variable has skewness parameter different from \(\pm 1\) then its cdf is positive on whole \(\mathbb{R}\), the pdf’s \(f_{X_1}, \ldots, f_{X_d}\) and \(f_{X_0}\) are positive on whole \(\mathbb{R}\), the cdfs \(F_{X_1}, \ldots, F_{X_d}\) are strictly increasing. Then the functions \(T_k: \mathbb{R} \rightarrow \mathbb{R}, k = 1, \ldots, d\), defined by

\[
T_k(u) = F_{X_k}^{-1}(F_{X_0}(u)),
\]

are strictly increasing functions. Besides,

\[
T_k'(u) = f_{X_k}(u)/f_{X_0}(T_k(u)), \ {k = 1, \ldots, d}
\]

are positive functions. This implies \(f_{X_0}(T_k(u))T_k'(u)du = f_{X_k}(u)du\), that yields

\[
F_{X_k}(T_k(u)) = \int_{-\infty}^{T_k(u)} f_{X_0}(T_k(u))T_k'(u)du = \int_{-\infty}^{u} f_{X_k}(u)du = F_{X_k}(u).
\]

On the other hand, for every \(t \in \mathbb{R}\) we have
Continuing the above process to the second step, we change the coordinates of \( e_d = (0;0,0,\ldots,1) \) by moving the first coordinate to the end and shifting the others ahead by one place (that means the basis \((e_d,e_{d-1},\ldots,e_2,e_1)\) in \( \mathbb{R}^d \) is replaced by the basis \((e_2,e_3,\ldots,e_{d-1},e_d)\). With the new rv, we get
\[
(R,\Theta_1^{(i)},\ldots,\Theta_{d-1}^{(i)}) = B \left( Z_1^{(i)}, Z_2^{(i)}, \ldots, Z_{d-1}^{(i)} \right).
\]
The second transformation \( t^{(2)}(Z^{(1)}) = Z^{(2)} \) is defined by
\[
Z^{(2)} = \left( Z_1^{(2)}, Z_2^{(2)}, \ldots, Z_d^{(2)} \right) = D \left( R, \Theta_1^{(i)}, \ldots, \Theta_{d-2}^{(i)}, 2\pi F_{\Theta_{d-1}^{(i)}} \circ \Theta_{d-1}^{(i)} \right).
\]
The random variable \( 2\pi F_{\Theta_{d-1}^{(i)}} \circ \Theta_{d-1}^{(i)} \) is uniformly distributed on \([0;2\pi]\), then by a similar argument as the above, we confirm that the distribution of the rv \( Z^{(2)} \) is symmetric and invariant under all rotations around the origin in \( \mathcal{P}_{(e_d,\ldots,e_2,e_1)} \). Simultaneously, \( Z^{(2)} \) is also invariant under all rotations around the origin in \( \mathcal{P}_{(e_d,\ldots,e_2,e_1)} \) because \( Z^{(1)} \) is invariant under all rotations around the origin in \( \mathcal{P}_{(e_d,\ldots,e_2,e_1)} \).

Continuing the above process to the \( d \)-th step, the basis \((e_d,e_{d-1},\ldots,e_{d-3},e_{d-2})\) is replaced by the basis \((e_d,e_{d-1},\ldots,e_{d-3},e_{d-2})\), the rv obtained after the \((d-1)\)-th step \( (Z_1^{(d-2)}, Z_2^{(d-2)}, \ldots, Z_{d-1}^{(d-2)}, Z_d^{(d-2)}) \) is rearranged into the rv \( (Z_2^{(d-2)}, Z_3^{(d-2)}, \ldots, Z_d^{(d-2)}, Z_1^{(d-2)}) \). Then the polar representation of the new rv is
\[
B \left( Z_2^{(d-2)}, Z_3^{(d-2)}, \ldots, Z_d^{(d-2)}, Z_1^{(d-2)} \right) = (R, \Theta_1^{(d-2)}, \ldots, \Theta_{d-1}^{(d-2)}, \Theta_{d-2}^{(d-2)}).
\]
By applying \( D \) to the polar representation rv replaced the last coordinate \( \Theta_{d-1}^{(d-2)} \) by \( 2\pi F_{\Theta_{d-1}^{(d-2)}} \circ \Theta_{d-1}^{(d-2)} \), we have \( U^{(d-1)}(Z^{(d-2)}) = Z^{(d-1)} \), where
\[
Z^{(d-1)} = \left( Z_1^{(d-1)}, Z_2^{(d-1)}, \ldots, Z_d^{(d-1)} \right) \sim D \left( R, \Theta_1^{(d-2)}, \ldots, \Theta_{d-2}^{(d-2)}, 2\pi F_{\Theta_{d-1}^{(d-2)}} \circ \Theta_{d-1}^{(d-2)} \right).
\]
For the same reason as presented above, we can conclude the distribution of the rv \( Z^{(d-1)} \) has symmetric distribution and is invariant under every two-dimensional rotation around the origin in \( \mathcal{P}_{(e_{d-2},\ldots,e_2,e_1)} \). Consequently, the distribution is invariant under every two-dimensional rotation around the origin in each of the two-dimensional planes \( \mathcal{P}_{(e_{d-2},\ldots,e_2,e_1)} \). This confirms the fact that the distribution of the rv \( Z^{(d-1)} \) is invariant under all linear rotations around the origin \( 0 \) in the whole \( \mathbb{R}^d \), which means the rv has isotropic distribution.

Let \( V(x_d,x_{d-1},\ldots,x_2,x_1) \) and define \( U = V \circ U^{(d-1)} \circ U^{(d-2)} \circ \ldots \circ U^{(2)} \circ U^{(1)} \). Then we see all the transformations \( U^{(1)}, U^{(2)}, \ldots, U^{(d-2)}, U^{(d-1)} \) are essentially based on the random variables defined in \( \mathbb{R}^{d-1} \). That implies the existence of an invertible differentiable transformation \( Q: \mathbb{R}^{d-1} \rightarrow \mathbb{R}^{d-1} \) such that \( U = D \circ Q \circ B \). Besides, the above argument ensures that the rv \( Y = U(Z) = U(T(X)) \) is an isotropic rv. This together with Lemma 2.1 and Lemma 2.2 show \( Y \) is a sub-Gaussian rv. Therefore, we can confirm \( K = U \circ T \) is the desired transformation. \( \square \)

Theorem 2.3 ensures \( Y = K(X) = (A_1^{1/2}G_1,\ldots,A_1^{1/2}G_d) \) is a sub-Gaussian rv, where \( G = (G_1,\ldots,G_d) \) is a zero-mean Gaussian vector in \( \mathbb{R}^d \) independent of \( A \), with iid components \( G_i \), \( i = 1,\ldots,d \), and \( A \sim \mathcal{S}(\alpha/2;1;\{\cos(\pi/4)\}^{1/2};0) \). We get \( R^2 = A_1^2 + \ldots + A_d^2 = AH \), where \( H \) is chi-squared with \( d \) degrees of freedom, independent of \( A \), and \( R = \|Y\| = (Y_1^2 + \ldots + Y_d^2)^{1/2} \). Then the density function of \( R \) (see [5] [16]) can be expressed as
\[ f_A (r) = 2r \int_0^r f_A \left( \frac{r^2}{t} \right) t^2 dt, \]

where \( f_A \) and \( f_H \) are density functions of random variables \( A \) and \( H \), respectively. Further, the argument of Subsection 2.1 [16] yields the following.

**Lemma 2.4.** The density function of \( Y \) has the following form

\[ f_Y (y) = \begin{cases} \frac{\Gamma(d/2)}{(2\pi^d)} |y|^{\frac{d-2}{2}} f_B (\|y\|) & y \neq 0 \\ \frac{\Gamma(d/2)}{(\alpha^2 d/2 \pi^{d/2})} \Gamma(d/2) |y|^{\frac{d-2}{2}} & y = 0, \end{cases} \]

with the density function \( f_B \) given in (11).

Due to \( Y = K(X) = U(T'(X)) \), it is clear that

\[ f_Y (x) = J_U (x) \times J_r (x) \times f_K (K(x)), \]

where \( J_U \) and \( J_r \) are respectively the Jacobian's of the operators \( U \) and \( T = (T_1, ..., T_d) \) defined in the proof of Theorem 2.3. Meanwhile, it is easy to see

\[ J_r (x_1, ..., x_d) = \frac{f_K (x_1) \times...\times f_K (x_d)}{f_{S(\alpha,0,0,0)} (F'_{X_1}(x_1)) \times...\times f_{S(\alpha,0,0,0)} (F'_{X_d}(x_d))}, \]

and

\[ J_U (x_1, ..., x_d) = (2\pi)^d \times f_{\theta_{d-1}} \left( \frac{2 \arctan \frac{z_d}{z_{d-1} + \sqrt{z_{d-1}^2 + z_d^2}}}{z_d + \sqrt{z_{d-1}^2 + z_d^2}} \right) \times...\times f_{\theta_{d-3}} \left( \frac{2 \arctan \frac{z_{d-2}}{z_{d-2} + \sqrt{z_{d-2}^2 + z_{d-1}^2}}}{z_{d-2} + \sqrt{z_{d-2}^2 + z_{d-1}^2}} \right), \]

where \( z_k = F^{-1}_{S(\alpha,0,0,0)} (F'_{X_k}(x_k)), \) for \( k = 1, ..., d, \) and \( \theta_{d-1}, \theta_{d-2}, ..., \theta_{d-2} \) are defined in the proof of Theorem 2.3. The equations (12) - (15) together with Theorem 2.3 provide the following theorem.

**Theorem 2.5.** Let \( X \) be a given stable rv in \( \mathbb{R}^d \). Suppose the skewness parameters of all marginals of \( X \) are different from \( \pm 1 \). Then the probability density function of \( X \) can be expressed as

\[ f_X (x) = \begin{cases} h(x) \cdot g(x) \cdot f_B (g(x)) \times \frac{\Gamma(d/2)}{2\pi^{d/2}} & x \neq (\text{med}_1, ..., \text{med}_d) \\ \Gamma(d/2)/\alpha^{2d/\alpha \pi^{d/2}} & x \neq (\text{med}_1, ..., \text{med}_d) \end{cases} \]

where the density function \( f_B \) given in (11),

\[ g(x) = \sqrt{\left[ F_{S(\alpha,0,0,0)} (F'_{X_1}(x_1)) \right]^2 + ... + \left[ F_{S(\alpha,0,0,0)} (F'_{X_d}(x_d)) \right]^2}. \]

for \( x = (x_1, ..., x_d) \in \mathbb{R}^d, h(x) = J_U (x) \times J_r (x), \) with \( J_U (x) \) and \( J_r (x) \) expressed in (14) and (15).

**Proof.** It is evident in the proof of Theorem 2.3 that the rv \( Z = T(X) \) is stable with symmetric marginals \( S(\alpha; 0; 1; 0) \)-distributed. Besides,

\[ T(\text{med}_1, ..., \text{med}_d) = (0, ..., 0). \]

Meantime, Theorem 2.3 ensures the rv \( Y = U(Z) \) to be a sub-Gaussian rv. Therefore, Lemma 2.4 provides the probability density function \( f_Y \) of \( Y \) expressed as (12). Simultaneously, it is clear that the transformation \( U \) does not change the amplitude of any vector \( x \in \mathbb{R}^d, \) that confirms

\[ \|y\| = \|f'(x)\| = \left[ \left( F_{S(\alpha,0,0,0)} (F'_{X_1}(x_1)) \right)^2 + ... + \left( F_{S(\alpha,0,0,0)} (F'_{X_d}(x_d)) \right)^2 \right]^{1/2}. \]

For \( x = (x_1, ..., x_d) \). This together with (13) - (15) and (17) yield (16). \( \square \)

### III. TEST ON THE STABLE DISTRIBUTION OF MULTIVARIATE DATA

Let \( x = (x_i; i = 1, 2, ..., d; j = 1, 2, ..., n) \) be a dataset collected from a rv \( X = (X_1, ..., X_d) \). The transformation \( K \) in Theorem 2.3 turns \( x \) into \( y = K(x) = (y_{ij}; i = 1, 2, ..., d; j = 1, 2, ..., n), \) that can be considered as a dataset extracted from the rv \( Y = (Y_1, ..., Y_d) = K(X) \). According to Theorem 2.3, the rv \( X \) has stable distribution with all marginal skewness parameters different from \( \pm 1 \) iff \( Y \) is a sub-Gaussian rv. Simultaneously, in spirit of (9), it is easy to see that the rv \( Y \) is a sub-Gaussian rv iff its copula is the copula of sub-Gaussian rv and all its marginals have \( \alpha \)-stable distribution with common stable index \( \alpha \). Then the stable distribution of \( X \) can be checked by conducting the following two actions.

- a) Use the non-parametric test based on the Kendall functions ([13], [20], [21]) to check the goodness-of-fit of the copula of \( Y \) to sub-Gaussian copula;
- b) Use the Kolmogorov - Smirnov test to check the goodness-of-fit of all marginals \( Y_1, ..., Y_d \) to \( \alpha \)-
stable distribution \( S(\alpha, \beta; 0; 1; 0) \), where \( \alpha_0 = (\alpha_1 + \alpha_2 + \ldots + \alpha_d)/d \), and \( \alpha_i \) is the estimated stable index of the data marginal \( x_i^j = \{x_{ij}; j = 1, \ldots, n\} \) for \( i = 1, \ldots, d \).

The functions `McCullochParametersEstim` and `ks.test` in the R software package can be used to estimate the stable parameters of each data marginal \( x_i^j = \{x_{ij}; j = 1, \ldots, n\} \), \( i = 1, \ldots, d \), and then to verify the hypothesis of stable distribution goodness-of-fit for all marginals \( i = 1, \ldots, d \) in the task b). Meanwhile, the task a) can be completed by conducting the following steps.

**Step 1.** Generate a hypothetical dataset \( y^{(0)} = \{y_{ij}^{(0)}; i = 1, 2, \ldots, d; j = 1, 2, \ldots, n\} \) that is the dataset extracted from the rv with sub-Gaussian distribution \( Y = (Y_1^1, Y_2^1, \ldots, Y_d^1) \), where \( G \sim N(0; I) \) is a Gaussian rv with expectation \( \mathbf{0} \) and covariance matrix \( \mathbf{I} \), the unit matrix of the size \( d \times d \), \( A \sim \mathcal{N}(\mathbf{0}; 1; I) \) and \( A \) and \( G \) are independent.

**Step 2.** Let \( M_k = \#\{j \neq k; y_{1k}^{(0)} < y_{1j}^{(0)}, y_{2k}^{(0)} < y_{2j}^{(0)}, \ldots, y_{dk}^{(0)} < y_{dj}^{(0)}\}/n \) for \( k = 1, 2, \ldots, n \). Based on the formula of the Kendall functions \([23]\) of the rv

\[ K_\chi(t) = P\{F_\chi^- \{y_{1k}^{(0)}, y_{2k}^{(0)}, \ldots, y_{dk}^{(0)}\} \leq t\}, \]

to estimate the values of the Kendall function at \( M_k \) for the hypothetical dataset

\[ K_{\chi}^{(0)}(M_k) = \#\{j; F_\chi^- \{y_{ij}^{(0)}, y_{1j}^{(0)}, \ldots, y_{dj}^{(0)}\} \leq M_k\}/n, \]

and for the transformed sample dataset

\[ K_{\chi}(M_k) = \#\{j; F_\chi^- \{y_{ij}^{(0)}, y_{1j}^{(0)}, \ldots, y_{dj}^{(0)}\} \leq M_k\}/n, \]

where \( F \) denotes the empirical distribution function. To determine the test statistic

\[ d = \sum_{k=1}^n \left(K_{\chi}^{(0)}(M_k) - K_{\chi}(M_k)\right)^2, \]

**Step 3.** Apply the Monte-Carlo sampling procedure by repeating Step 1 to construct 1000 hypothetical datasets \( y^{(m)} = \{y_{ij}^{(m)}; i = 1, 2, \ldots, d; j = 1, 2, \ldots, n\}, m = 1, 2, \ldots, 1000 \). Then repeatedly conduct Step 2 with the transformed sample dataset \( y \) replaced by each of the new obtained hypothetical datasets to get the values \( d_m = \sum_{k=1}^n (K_{\chi}^{(0)}(M_k) - K_{\chi}(M_k))^2 \). With a given probability value \( p \in (0; 1) \), the critical value \( L_p \) of test for the hypothesis \( H \) “\( Y \) has sub-Gaussian copula” is taken equal to the \((1-p)\)-quantile of the set \( \{d_m, m = 1, 2, \ldots, 1000\} \). Reject \( H \) if \( d \geq L_p \) and accept \( H \) if \( d < L_p \).

In the following we present two examples of the application of the above goodness-of-fit test procedure to examine the multivariate stable distribution of the daily return data of Nasdaq Finance. The daily return data are considered, i.e. daily returns are measured by the log compounded percentage returns to examine the multivariate stable distribution of the daily return data of Nasdaq Finance. The daily return data of the set \( \{\text{NFLX}; \text{ZM}; \text{AMC}; \text{AXP}; \text{EJ}\} \), the function `McCullochParametersEstim` in the R software package is used to estimate the stable parameters of each coordinate, giving the results in Table 1.

<table>
<thead>
<tr>
<th>Coordinate</th>
<th>( \alpha )</th>
<th>( \alpha_1 )</th>
<th>( \beta )</th>
<th>( \gamma )</th>
<th>( \delta )</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_1 ) (NFLX)</td>
<td>1.771</td>
<td>1.478</td>
<td>-0.149</td>
<td>1.550</td>
<td>0.006</td>
<td>0.5699</td>
</tr>
<tr>
<td>( X_2 ) (ZM)</td>
<td>1.461</td>
<td>1.478</td>
<td>0.000</td>
<td>2.090</td>
<td>-0.169</td>
<td>0.3648</td>
</tr>
<tr>
<td>( X_3 ) (AMC)</td>
<td>1.392</td>
<td>1.478</td>
<td>0.046</td>
<td>2.719</td>
<td>-0.264</td>
<td>0.6843</td>
</tr>
<tr>
<td>( X_4 ) (AXP)</td>
<td>1.388</td>
<td>1.478</td>
<td>-0.152</td>
<td>1.128</td>
<td>0.029</td>
<td>0.6843</td>
</tr>
<tr>
<td>( X_5 ) (FA)</td>
<td>1.378</td>
<td>1.478</td>
<td>0.134</td>
<td>1.229</td>
<td>1.229</td>
<td>0.5699</td>
</tr>
</tbody>
</table>

The function `ks.test` is used to verify the hypotheses of stable distribution goodness-of-fit \( X_i \sim S(\alpha_i; \beta_i; \gamma_i; \delta_i), i = 1, \ldots, 5 \), with \( \alpha = (\alpha_1 + \ldots + \alpha_5)/5 = 1.478 \), giving the p-values 0.5699; 0.3648; 0.6843; 0.6843; and 0.5699, indicate all coordinate data fit to the stable distributions with the corresponding parameters.

Then we conduct Step 1 followed by Step 2 and Step 3 to get the test statistic \( d = 26.4866 \) and the test critical values 87.6010; 126.6208; 188.3399; 236.3742; and 302.8049, corresponding to the significance levels 0.10; 0.05; 0.01; 0.005; and 0.001, respectively. Because the test statistic \( d = 26.4866 \) is smaller than the critical values, the hypothesis stating that the transformed rv \( Y = k(X) \) has sub-Gaussian copula is accepted. Simultaneously, the R function `ks.test` gives the p-values 0.4115; 0.3648; 0.8900; 0.8455; and

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showing the coordinate \( y_1, y_2, y_3, y_4 \) and \( y_5 \) of the transformed data \( y = K(x) \) fit to the stable distribution \( S(1.478;0;1;0) \). The above two facts confirm \( Y \) is a sub-Gaussian rv. Consequently, Theorem 2.3 allows to conclude that the returns’ dataset of NFLX, ZM, AMC, AXP, and FA fits to the 5-dimensional 1.478-stable distribution.

**Example 2.** Regarding the 5-dimensional data of the stocks JNJ; XOM; FB; HD; and PPG, we proceed with the same procedure as that of Example 1. The function McCullochParametersEstim gives the estimated stable parameters in Table II.

| TABLE II: STABLE PARAMETERS OF THE MARGINALS JNJ, XOM, FB, HD, PPG |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| Coordinate     | \( \alpha \)    | \( \alpha_i \)  | \( \beta \)     | \( \gamma \)    | \( \delta \)    |
| \( X_1 \) (JNJ) | 1.385           | 1.4444          | -0.313          | 0.642           | 0.098           | 0.2462          |
| \( X_2 \) (XOM) | 1.329           | 1.4444          | 0.061           | 1.146           | -0.129          | 0.1847          |
| \( X_3 \) (FB)  | 1.580           | 1.4444          | -0.264          | 1.261           | 0.127           | 0.2821          |
| \( X_4 \) (HD)  | 1.465           | 1.4444          | -0.150          | 0.808           | 0.110           | 0.6843          |
| \( X_5 \) (PPG) | 1.463           | 1.4444          | -0.003          | 0.953           | 0.083           | 0.8900          |

The \( R \) function \( ks.test \) is used to verify the hypotheses of stable distribution goodness-of-fit \( X_i \sim S(\alpha_i; \beta_i; y_i; \delta_i), \; i = 1, \ldots, 5, \) with \( \alpha = (\alpha_1 + \ldots + \alpha_5)/5 = 1.4444 \), giving the \( p \)-values 0.2462; 0.1847; 0.2821; 0.6843; and 0.8900, confirming the \( \alpha \)-stability of all marginal distributions of the concerned data.

Then Step 1, Step 2, and Step 3 of the proposed procedure are sequentially proceeded to provide the test statistic \( d = 342.6804 \) and the test critical values 115.2128; 131.7771; 168.7637; 173.2055; and 186.3036 of the test on sub-Gaussian 1.4444-stable copula in ~ \( ^* \), corresponding to the significance levels 0.10; 0.05; 0.01; 0.005; and 0.001, respectively. Since the test statistic \( d \) is greater than all the critical values, we can realize that the hypothesis stating that the transformed rv \( Y = K(X) \) has sub-Gaussian copula is not accepted. This rejects the sub-Gaussian distribution of the transformed rv \( Y \). Therefore, Theorem 2.3 yields the conclusion that the returns’ dataset of JNJ, XOM, FB, HD, and PPG does not fit to 5-dimensional stable distribution, although all its marginals have stable distribution.

**IV. CONCLUSION AND DISCUSSION**

The results obtained in this study provide useful tools to investigate thoroughly multivariate data those have stable distribution. The bijective transformation in multidimensional space created on the base of Theorem 2.3, that turns a given stable random vector into a sub-Gaussian random vector, provides a theoretical basis of the testing procedure on the stable distribution of multivariate data. Examples of datasets collected from the NASDAQ stock market are used to illustrate the practicability of the proposed testing procedure. Theorem 2.5, an immediate consequence of Theorem 2.3, represents a method of probability density function estimation for multivariate data with stable distribution.

Let \( X \) be a random vector, its symmetrization is defined as \( X^* = X' - X^{**} \), where \( X' \) and \( X^{**} \) are two independent copies of \( X \). It is clear that if \( X \) is a stable random vector then \( X^* \) is also a stable random vector with all marginal skewness parameters equal to 0. Consequently, the rejection of stable distribution hypothesis for \( X^* \) implies the rejection of stable distribution hypothesis for \( X \). Therefore, although Theorem 2.3 is valid only for stable random vector with marginal’s having all skewness parameters different from \( \pm 1 \), the proposed stability testing procedure can be partially extended to the class of all stable distribution.

**REFERENCES**