Infinitely Many Weak Solutions for Problems Involving Both p(x)-Laplacian and p(x)-Biharmonic Operators

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Abstract

In this paper, we study the existence of infinitely many weak solutions for problems involving both p(x)-Laplacian and p(x)-Biharmonic operators. In the proof of our main result, we use variational methods and the known symmetric mountain pass lemma.

Index Terms

Existence result, Variational methods, Eigenvalues, p(x)-biharmonic operator, p(x)-Laplacian operator, Sobolev space, Symmetric mountain pass lemma.

I. Introduction

OVER the last decade, the study of variational problems with operators with variable exponents growth conditions have sparked intense research efforts. Such problems which often called nonhomogeneous eigenvalue problems has a special feature that has a prossesses more complicated to use many of the techniques that are used in the homogeneous case (when the exponent is a positive constant). This type of problems has various significant applications in many topics like elastic mechanics [1], electrortheological fluids [2], image processing model [3], [4], stationary thermorheological viscous flows [3] and the mathematical description of the processes filtration of an idea barotropic gas through a porous medium (see [4]).

In the present work, we are interested with the existence of the weak solutions for the following nonhomogeneous eigenvalue problem:

$$(P_\lambda) : \begin{cases} \Delta^2_{p(x)} - \Delta_{p(x)} - a(x)|u|^{\alpha(x)-2}u = \lambda(b_1(x)|u|^{\beta(x)-2}u - b_2(x)|u|^{\gamma(x)-2}u), & \text{in } \Omega \\ u \in W^{2,p(.)}(\Omega) \cap W_0^{1,p(.)}(\Omega), \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary, λ is a positive real, p, α, β and γ are continuous functions on $\overline{\Omega}$, $\Delta_{p(x)} = div(|\nabla u|^{p(x)-2}\nabla u)$ is the p(x)-Laplacian operator, $\Delta^2_{p(x)}u = \Delta(|\Delta u|^{p(x)-2}\Delta u)$, is the fourth order differential operator called $p(x)$-biharmonic, $a \in C(\overline{\Omega})$, $b_1 \in L^{s_1(x)}$ and $b_2 \in L^{s_2(x)}$ with $s_1, s_2 \in C_+(\Omega)$ and a, b_1 and b_2 are nonnegative functions.

Special cases of problem (P_λ) have been treated in recent years by several authors. The case $a(x) \equiv b_2(x) \equiv 0$ and $p(x) \equiv \beta(x) \equiv const$ has been investigated by El Khalil, Kellati and Touzani [5], who have shown the existence of an increasing sequence of strictly positive eigenvalues using the Ljusternich-schnirelmann theory on varieties of class C^1. By using the same theory and the Ekelands variational principale, the particular case $a(x) \equiv b_2(x) \equiv 0$ and $b_1(x) \equiv 1$ have been studied by Ayoujil and Amrouss [6] in tree cases as follow:

- For the case $\underset{x\in\overline{\Omega}}{max} : \beta(x) < \underset{x\in\overline{\Omega}}{min}\, p(x)$, the authors proved that the energy function associated to the problem (P_λ) has a non-trivial minimum for any strictly positive λ.
- The case when $\underset{x\in\overline{\Omega}}{min}\, \beta(x) < \underset{x\in\overline{\Omega}}{min}\, p(x)$ and $\beta(x)$ has a subcritical subcritical growth, they used the principle in order to prove the existence of a continuous sequence of eigenvalues continuous sequence of eigenvalues in a neighborhood of the origin.
- The case when $\underset{x\in\overline{\Omega}}{max}\, p(x) < \underset{x\in\overline{\Omega}}{min}\, \beta(x) \leq \underset{x\in\overline{\Omega}}{max}\, \beta(x) < \frac{Np(x)}{N-2p(x)}$ the authors have shown, by the mountain pass theorem, that for any $\lambda > 0$, the corresponding energy function at the critical points of (P_λ) is non-trivial and positive, and hence the set of eigenvalue $\Lambda = (0, +\infty)$.

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For the case $p(x)=\beta(x)$, we refer again to Ayoujil and Amrouss, who studied the spectrum of (P_λ) . They showed that Λ contains an increasing sequence of positive eigenvalues $(\lambda_k)_{k\geq 1}$ such that $\sup\Lambda=+\infty$. Moreover, contrary to the case $p\equiv Const$, they proved that $inf\Lambda=0$, under some special conditions related to the monotonicity of the function $p(x)$.

lastly, in [7], [8], the authors give sufficient conditions to prove the existence of at least one nontrivial weak solution for a similar problem where the expression "$\Delta(|\Delta u|^{p(x)-2}\Delta u)+a(x)|u|^{p(x)-2}u$" appears instead of the expression on the left side of the equation of (P_λ). More recently, Laghzal and Touzani [9] proved that the following problem

$$\begin{cases}\Delta^2_{p(x)}-\lambda a(x)|u|^{\alpha(x)-2}u=\mu b(x)|u|^{\beta(x)-2}u, & \text{in } \Omega\\ u\in W^{2,p(.)}(\Omega)\cap W_0^{1,p(.)}(\Omega),\end{cases}$$

has infinitely many eigengraph sequences. This was done using the Ljusternik-Schnirelmann critical point theory.

Inspired by the above-mentioned papers and based on the symmetric mountain pass lemma [10], we study the problem (P_λ) in this new situations. More precisely, we will show that our problem admits a sequence of nontrivial weak solutions.

The paper is structured as follows: In the next section, we first recall some basic notions and properties used in the paper. The section 3 is devoted to the existence of infinitely many weak solutions of the problem (P_λ). Comparison with some new existing results are included in the final part of this paper.

II. Preliminaries

In this section, we recall some results on the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ (for details, see [11], [12], [13]) and some properties of the $p(x)$-biharmonic operator, which will be needed later. In the sequel, X will denote the function space $W^{2,p(x)}(\Omega)\cap W_0^{1,p(x)}(\Omega)$ and $C_+(\overline{\Omega})$ will denote the set defined by

$$C_+(\overline{\Omega}):=\{h:h\in C(\overline{\Omega}),h(x)>1,\ \text{for all } x\in\overline{\Omega}\}.$$

Let p be a Lipschitz continuous function on $\overline{\Omega}$, we denote by $1<p^-:=\min_{x\in\overline{\Omega}}p(x)\leq p^+=\max_{x\in\overline{\Omega}}p(x)<\infty$ and

$$L^{p(x)}(\Omega)=\{u:\Omega\to\mathbb{R}\text{ measurable such that }\int_\Omega|u(x)|^{p(x)}dx<\infty\}.$$

We recall also the following so-called Luxemburg norm on this space defined by the formula

$$|u|_{p(x)}=\inf\{\nu>0:\int_\Omega\left|\frac{u(x)}{\nu}\right|^{p(x)}dx\leq 1\}.$$

Clearly, when $p(x)=p$, a positive constant, the space $L^{p(x)}(\Omega)$ is the classical Lebesgue space $L^p(\Omega)$ and the norm $|u|_{p(x)}$ is the standard one $\|u\|_{L^p}=\left(\int_\Omega|u|^pdx\right)^{\frac{1}{p}}$ in $L^p(\Omega)$.
For any positive integer k, as in the constant exponent case, let

$$W^{k,p(x)}(\Omega)=\{u\in L^{p(x)}(\Omega):D^\alpha u\in L^{p(x)}(\Omega),|\alpha|\leq k\},$$

here $\alpha=(\alpha_1,...,\alpha_N)$ is a multi-index, $|\alpha|=\sum_{i=1}^N\alpha_i$ and $D^\alpha u=\frac{\partial^{|\alpha|}u}{\partial^{\alpha_1}x_1...\partial^{\alpha_N}x_N}$. Then, $W^{k,p(x)}(\Omega)$ is a separable and reflexive Banach space equipped with the norm

$$\|u\|_{k,p(x)}=\sum_{|\alpha|\leq k}|D^\alpha u|_{p(x)}.$$

$W_0^{k,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$.

Proposition II.1. *(Hölder inequality) Suppose that $u\in L^{p(x)}(\Omega)$ and $v\in L^{p'(x)}(\Omega)$ with $L^{p'(x)}$ is the dual space of $L^{p(x)}(\Omega)$ (where p' is the conjugate function of p, i.e., $p'(x)=\frac{p(x)}{p(x)-1}$ for all $x\in\Omega$. Then we have the following inequality*

$$\int_\Omega|uv|dx\leq\left(\frac{1}{p^-}+\frac{1}{p'^-}\right)|u|_{p(x)}|v|_{p'(x)}\leq 2|u|_{p(x)}|v|_{p'(x)}.$$

Furthermore, if p_1, p_2, $p_3\in C_+(\Omega)$ and $\frac{1}{p_1(x)}+\frac{1}{p_2(x)}+\frac{1}{p_3(x)}=1$, then for all $u\in L^{p_1(x)}(\Omega),\ v\in L^{p_2(x)}et\ w\in L^{p_3(x)}$ we have the following inequality (see [14, Proposition 2.5]):

$$\int_\Omega|uvw|dx\leq\left(\frac{1}{p_1^-}+\frac{1}{p_2^-}+\frac{1}{p_3^-}\right)|u|_{p_1(x)}|v|_{p_2(x)}|w|_{p_3(x)},\tag{1}$$

Proposition II.2. *Let* $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$ *defined by:*

$$\rho(u) = \int_\Omega |u|^{p(x)} dx.$$

Then we have the following relations:

$$\begin{aligned} |u|_{p(x)} < 1 (= 1, > 1) &\Leftrightarrow \rho(u) < 1 (= 1, > 1), \\ |u|_{p(x)} > 1 &\Rightarrow |u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}, \\ |u|_{p(x)} < 1 &\Rightarrow |u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}, \\ |u_n - u|_{p(x)} \to 0 &\Leftrightarrow \rho(u_n - u) \to 0. \end{aligned}$$

We also recall the following proposition which will be necessary later on:

Proposition II.3. *[15] Let* p *et* q *be measurable functions such that* $p \in L^\infty(\Omega)$ *et* $1 < p(x)q(x) \leq \infty$*, for all* $x \in \Omega$*. Let* $u \in L^{q(x)}(\Omega), u \neq 0$*. Then we have*

$$\begin{aligned} |u|_{p(x)} \leq 1 &\Rightarrow |u|_{p(x)q(x)}^{p^+} \leq \left| |u|^{p(x)} \right|_{q(x)} \leq |u|_{p(x)q(x)}^{p^-}, \\ |u|_{p(x)} \geq 1 &\Rightarrow |u|_{p(x)q(x)}^{p^-} \leq \left| |u|^{p(x)} \right|_{q(x)} \leq |u|_{p(x)q(x)}^{p^+}. \end{aligned} \tag{2}$$

Theorem II.4. *[16] Let* $p \in C_+(\Omega)$ *and*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-2p(x)}; & p(x) < \frac{N}{2} \\ +\infty, & p(x) \geq \frac{N}{2}. \end{cases}$$

If $q \in C_+(\Omega)$ *such that* $q(x) < p^*(x)$*, then there is a continuous and compact injection from* X *to* $L^{q(x)}(\Omega)$*.*

Lemma II.5. *[17]* $\Delta^2_{p(.)} : W_0^{2,p(.)}(\Omega) \to W_0^{-2,p'(.)}(\Omega)$ *is a mapping of type* (S_+)*, that is, if*

$$u_n \rightharpoonup u \text{ in } W_0^{2,p(.)}(\Omega) \quad \text{and} \quad \underset{n\to\infty}{limsup} \langle \Delta^2_{p(.)} u_n, u_n - u \rangle \leq 0,$$

then $u_n \to u$ *in* $W_0^{2,p(.)}(\Omega)$ *.*

Lemma II.6. *[14]* $-\Delta_{p(.)} : W_0^{1,p(.)}(\Omega) \to W_0^{-1,p'(.)}(\Omega)$ *is a mapping of type* (S_+)*.*

Note that weak solutions of problem (P_λ) are considered in the generalized Sobolev space $X := W^{2,p(.)}(\Omega) \cap W_0^{1,p(.)}(\Omega)$, endowed with the norm

$$\|u\|_{p(.)} = |\Delta u|_{p(.)} + |\nabla u|_{p(.)}.$$

We note also that $\|u\|_{p(.)}$ and $|\Delta u|_{p(.)}$ are two equivalent norms of X.

Let

$$\|u\| = inf \left\{ \kappa > 0 : \int_\Omega \left[\left| \frac{\Delta u(x)}{\kappa} \right|^{p(x)} + \left| \frac{\nabla u(x)}{\kappa} \right|^{p(x)} \right] dx \leq 1 \right\}.$$

Then, $\|.\|$ is equivalent to the norms $\|u\|_{p(.)}$ and $|\Delta u|_{p(.)}$ in X.

We consider for all $u \in X$ the functional

$$\Lambda_p(u) = \int_\Omega (|\Delta u(x)|^{p(x)} + |\nabla u(x)|^{p(x)}) dx,$$

and give the following fundamental proposition.

Proposition II.7. *Let* $u \in X$*. the following relations hold*

- $\|u\|_{p(x)} < 1 (= 1, > 1) \Leftrightarrow \Lambda_p(u) < 1 (= 1, > 1)$,
- $\|u\|_{p(x)} > 1 \Rightarrow \|u\|_{p(x)}^{p^-} \leq \Lambda_p(u) \leq \|u\|_{p(x)}^{p^+}$,
- $\|u\|_{p(x)} < 1 \Rightarrow \|u\|_{p(x)}^{p^+} \leq \Lambda_p(u) \leq \|u\|_{p(x)}^{p^-}$,
- $\|u_n\| \to 0 \Leftrightarrow \Lambda_p(u_n) \to 0$,
- $\|u_n\| \to \infty \Leftrightarrow \Lambda_p(u_n) \to \infty$.

The proof of this proposition is similar to the proof of [18, Theorem 1.3].

Definition II.8. *A point* u_0 *of* X *is said to be a critical point of* ψ *if* $\psi'(u_0) = 0$*; the corresponding value* $c = \psi(u_0)$ *is called critical value of* ψ*.*

Definition II.9. *In a Banach space X, a sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that: $\psi(u_n) \to c$ in $\mathbb{R}$ and $\psi'(u_n) \to 0$ in X^* the dual of X, is called a Palais-Smale sequence at c level.*

Definition II.10. *(The Palais-Smale condition)*

Let X be a Banach space and $\psi : X \to \mathbb{R}$ a functional of class $\mathcal{C}^1$. If $c \in \mathbb{R}$, we say that ψ checks the Palais-Smale condition at the level c if any sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that: $\psi(u_n) \to c$ in $\mathbb{R}$ and $\psi'(u_n) \to 0$ in X^* we can extract a sub-sequence which converges strongly in X to a critical point of ψ.

Corollary II.11. *Let X be a Banach space and $\psi \in C^1(X,\mathbb{R})$. Let ψ be bounded from below and verify the Palais-Smale (PS) condition at the c-level. Then ψ reaches its minimum c .*

Theorem II.12. *[10](Symmetrical mountain pass lemma)*
Let X be a Banach space of infinite dimension and $\psi \in C^1(X,\mathbb{R})$ a functional which satisfies the following two assumptions:

1) *$\psi(u)$ is even, bounded from below, $\psi(0) = 0$ and $\psi(u)$ satisfies the (PS) condition.*
2) *For every $k \in \mathbb{N}$, there exists $A_k \in \Gamma_k$ such that $\sup\limits_{u\in A_k} \psi(u) < 0$.*

Then, each $c_k := \inf_{A\in\Gamma_k} \sup_{u\in A} \psi(u)$ is a critical value of $\psi(u)$. Moreover, $\psi(u)$ admits a sequence of critical points u_k such that

$$\psi(u_k) < 0, \quad u_k \neq 0 \quad and \quad \lim_k u_k = 0,$$

where Γ_k is the family of closed symmetric subsets A of X such that $0 \notin A$ and $\gamma(A) \geq k$ with $\gamma(A)$ is the genus of A, i.e.,

$$\gamma(K) = inf\{k \in \mathbb{N} : \exists h : K \to \mathbb{R}^k \setminus \{0\} \text{ such that } K \text{ is compact}$$
$$\text{and } h \text{ is continuous and odd}\}.$$

III. Existence of a infinitely weak solutions

In this section, we will show the existence of infinitely many weak solutions of the problem (P_λ) based on the symmetric mountain pass lemma. To proceed, we assume in this section that $a \in C(\overline{\Omega})$, $b_1 \in L^{s_1(x)}$ and $b_2 \in L^{s_2(x)}$ with $s_1, s_2 \in C_+(\Omega)$ and that we have the following condition:

$$H(p,\alpha,\beta,\gamma,s) : 1 < \alpha(x) < \beta(x) < \gamma(x) < p(x) < \frac{N}{2} < s(x), \quad \forall x \in \Omega, \tag{3}$$

where $s(x) := \min\{s_1(x), s_2(x)\}$.

Let $s_1'(x)$ (resp. $s_2'(x)$) be the conjugate exponent of the function $s_1(x)$ (resp. $s_2'(x)$) and consider $\theta_1(x) := \frac{s_1(x)\beta(x)}{s_1(x)-\beta(x)}$ and $\theta_2(x) := \frac{s_2(x)\gamma(x)}{s_2(x)-\gamma(x)}$.

Definition III.1. *Let $u \in X$ be a fixed real. We say that u is a weak solution of problem (P_λ) if*

$$\int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta v dx + \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx + \int_\Omega a(x)|u|^{\alpha(x)-2}uv dx$$
$$-\lambda\Big(\int_\Omega b_1(x)|u|^{\beta(x)-2}uv dx - \int_\Omega b_2(x)|u|^{\gamma(x)-2}uv dx\Big) = 0,$$

for all $v \in X$.

Furthermore, if $u \in X \setminus \{0\}$ then we say that λ is the eigenvalue of problem (P_λ).

We consider the functionals $F, G, I, J, \psi_\lambda : X \to \mathbb{R}$ defined by:

$$F(u) = \int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx + \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx,$$

$$G(u) = \int_\Omega \frac{a(x)}{\alpha(x)}|u|^{\alpha(x)}dx,$$

$$I(u) = \int_\Omega \frac{b_1(x)}{\beta(x)}|u|^{\beta(x)}dx,$$

$$J(u) = \int_\Omega \frac{b_2(x)}{\gamma(x)}|u|^{\gamma(x)}dx.$$

and

$$\psi_\lambda(u) = F(u) - G(u) - \lambda F(u) + \lambda I(u).$$

Lemma III.2. $\psi_\lambda \in C^1(X,\mathbb{R})$.

Proof. It is sufficient to prove that $I \in C^1(X, \mathbb{R})$, that is, for any $h \in X$, one has

$$\lim_{t \to 0} \frac{I(u+th) - I(u)}{t} = \langle dI(u), h \rangle,$$

and $dI : X \to X^*$ is continuous.

Indeed, for all $h \in X$, we have

$$\begin{aligned}\lim_{t \to 0} \frac{I(u+th) - I(u)}{t} &= \frac{d}{dt} I(u+th)|_{t=0} \\ &= \left(\frac{d}{dt} \int_\Omega \frac{b_1(x)}{\beta(x)} |u+th|^{\beta(x)} dx \right)|_{t=0}.\end{aligned}$$

Now, we have the application $t \to \frac{b_1(x)}{\beta(x)} |u+th|^{\beta(x)}$ is differentiable for all $x \in \Omega$. Moreover, for $|t| < 1$, using the inequalities (1), (2) and the fact that

$$\frac{1}{s_1(x)} + \frac{1}{\frac{\beta(x)}{\beta(x)-1}} + \frac{1}{\theta_1(x)} = 1,$$

we deduce that

$$\begin{aligned}\int_\Omega \frac{d}{dt} \left(\frac{b_1(x)}{\beta(x)} |u+th|^{q(x)} \right) dx &= \int_\Omega \left| V_1(x) |u+th|^{q(x)-2} (u+th) h \right| dx \\ &\leq \int_\Omega |b_1(x)| |u+th|^{\beta(x)-1} |h| dx \\ &\leq \int_\Omega |b_1(x)| (|u|+|h|)^{\beta(x)-1} |h| dx \\ &\leq c |V|_{s_1(x)} \, ||u+|h||_{\beta(x)}^{\beta^i - 1} \, |h|_{\theta(x)}, \qquad\qquad c > 0 \\ &< +\infty,\end{aligned}$$

where $i = +$ if $\left| |u|+|h| \right|_{\beta(x)} > 1$ and $i = -$ if $\left| |u|+|h| \right|_{\beta(x)} \leq 1$. Since $X \hookrightarrow L^{\theta_1(x)}(\Omega)$, $X \hookrightarrow L^{\beta(x)}(\Omega)$ and $b_1 \in L^{s_1(x)}(\Omega)$, then from the theorem of differentiation under the integral sign we deduce that

$$\begin{aligned}\lim_{t \to 0} \frac{I(u+th) - I(u)}{t} &= \int_\Omega \frac{d}{dt} \left(\frac{b_1(x)}{\beta(x)} |u+th|^{\beta(x)} \right) |_{t=0} dx \\ &= \int_\Omega b_1(x) |u+th|^{\beta(x)-2} (u+th) h |_{t=0} dx \\ &= \int_\Omega b_1(x) |u|^{\beta(x)-2} u h dx \\ &= \langle dI(u), h \rangle.\end{aligned}$$

Let us now show that $dI : X \to X^*$ is continuous.

For all $h \in X$ one has

$$\begin{aligned}|\langle dI(u), h \rangle| &= \left| \int_\Omega b_1(x) |u|^{\beta(x)-2} u h dx \right| \\ &\leq \int_\Omega |b_1(x)| |u|^{\beta(x)-1} |h| dx\end{aligned}$$

as $\frac{1}{\frac{\beta(x)}{\beta(x)-1}} = 1 - \frac{1}{\beta(x)} \leq 1 - \frac{1}{\beta^+} = \frac{\beta^+ - 1}{\beta^+}$ and according to Hölder's inequality we get

$$|\langle dI(u), h \rangle| \leq \left(\frac{1}{s_1^-} + \frac{\beta^+}{\beta^+ - 1} + \frac{1}{\theta_1^-} \right) |b_1|_{s_1(x)} \left| |u|^{\beta(x)-1} \right|_{\frac{\beta(x)}{\beta(x)-1}} |h|_{\theta_1(x)}.$$

Consequently by (2) we have

$$|\langle dI(u), h \rangle| \leq \left(\frac{1}{s_1^-} + \frac{\beta^+}{\beta^+ - 1} + \frac{1}{\theta_1^-} \right) |b_1|_{s_1(x)} |u|_{\beta(x)}^{\beta^i - 1} |h|_{\theta_1(x)}, \tag{4}$$

where $i = +$ if $|u|_{\beta(x)} > 1$ and $i = -$ if $|u|_{\beta(x)} \leq 1$.

From Lemma III.4, we know that $X \hookrightarrow L^{\theta_1(x)}(\Omega)$ is a compact injection. Therefore, there exists c_1 such that $|h|_{\theta_1(x)} \leq c_1\|h\|$. Therefore by (4) we deduce that

$$\begin{aligned}|dI(u), h\rangle| &\leq c_1\left(\frac{1}{s_1^-} + \frac{\beta^+}{\beta^+ - 1} + \frac{1}{\theta_1^-}\right)|b_1|_{s_1(x)}|u|^{\beta^i - 1}_{\beta(x)}\|h\| \\ &\leq c_2\|h\|,\end{aligned}$$

with $c_2 = c_1\left(\frac{1}{s_1^-} + \frac{\beta^+}{\beta^+-1} + \frac{1}{\theta_1^-}\right)|b_1|_{s_1(x)}|u|^{\beta^i-1}_{\beta(x)}$.

Thus $dI(u)$ is continuous, and since it is linear, we deduces that $dI(u) \in X^*$.

From the above we conclude that I is Fréchet differentiable, with

$$\langle I'(u), v\rangle = \int_\Omega b_1(x)|u|^{\beta(x)-2}uvdx$$

For all $u, v \in X$.

Therefore $I \in C^1(X, \mathbb{R})$.

Similarly, we show that $F, G, J \in C^1(X, \mathbb{R})$.

Hence $\psi_\lambda \in C^1(X, \mathbb{R})$. □

Remark III.3. *Since for all $u, v \in X$ one has*

$$\begin{aligned}\langle \psi'_{\lambda,}(u), v\rangle = &\int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta v dx + \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v dx - \int_\Omega a(x)|u|^{\alpha(x)-2}uvdx \\ &- \lambda\int_\Omega b_1(x)|u|^{\beta(x)-2}uvdx + \lambda\int_\Omega b_2(x)|u|^{\gamma(x)-2}uvdx,\end{aligned}$$

it is clear that u is a weak solution of (P_λ) if and only if u is a critical point of ψ_λ, i.e. $F'(u) - G'(u) = \lambda I'(u) - \lambda J'(u)$ in X^*.

Before presenting the main result, we will start with some results that we will need to prove the latter.

Lemma III.4. *The injections $X \hookrightarrow L^{\theta_1(x)}(\Omega)$ (resp. $X \hookrightarrow L^{\theta_2(x)}(\Omega)$) and $X \hookrightarrow L^{s'_1(x)\beta(x)}(\Omega)$ (resp. $X \hookrightarrow L^{s'_2(x)\gamma(x)}(\Omega)$) are compact and continuous.*

Proof. According to Theorem III.7 it is sufficient to show that $\theta_1(x) < p^*(x)$ and $s'_1(x)q(x) < p^*(x)$ $\left(with\ p^*(x) = \frac{Np(x)}{N-2p(x)}\right)$.

For the first inequality, by $H(p, \alpha, \beta, \gamma, s)$ we have $1 < \beta(x) < p(x) < \frac{N}{2} < s_1(x), \forall x \in \Omega$.

It is easy to show that

$$\frac{s_1(x)\beta(x)}{s_1(x) - \beta(x)} < \frac{\frac{N}{2}p(x)}{\frac{N}{2} - p(x)}.$$

Hence $\theta_1(x) < p^*(x)$.

For the second inequality we notice that

$$\frac{s_1(x)}{s_1(x) - 1}\beta(x) < \frac{s_1(x)}{s_1(x) - \beta(x)}\beta(x).$$

Hence

$$s'_1(x)\beta(x) < \theta_1(x).$$

Consequently

$$s'_1(x)\beta(x) < p^*(x).$$

□

Consider the following functionals defined as

$$\langle dF(u), v\rangle = \int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta v\, dx + \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v\, dx,$$

$$\langle dG(u), v\rangle = \int_\Omega a(x)|u|^{\alpha(x)-2}uv\, dx,$$

$$\langle dI(u), v\rangle = \int_\Omega b_1(x)|u|^{\beta(x)-2}uv\, dx,$$

and

$$\langle dJ(u), v\rangle = \int_\Omega b_2|u|^{\gamma(x)-2}uv\, dx.$$

Then we have the following two results which will be used later:

Proposition III.5. *dF satisfies the condition (S_+), that is, $u_n \rightharpoonup u$ in X and*

$$\limsup \langle dF(u_n), u_n - u\rangle \leq 0 \ \text{ implies } u_n \to u \text{ in } X.$$

Proof. Direct consequence of lemmas II.5 and II.6. □

Proposition III.6. *[6], [9] dG (resp. dI and dJ) is completely continuous, that is, $u_n \rightharpoonup u$ in X implies $dG(u_n) \to dG(u)$ in X' (resp. $dI(u_n) \to dI(u)$ and $dJ(u_n) \to dJ(u)$ in X').*

Now we can present our main result which will be proven using the symmetric mountain pass lemma.

Theorem III.7. *Suppose that the condition $H(p, \alpha, \beta, \gamma, s)$ defined in (3) holds and that $b_1 \in L^{s_1(x)}(\Omega)$ and $b_2 \in L^{s_2(x)}(\Omega)$ such that $b_1(x), b_2(x) > 0, \forall x \in \Omega$. Then for all $\lambda > 0$ there exists a sequence (u_n) of nontrivial weak solutions of problem (P_λ) such that $u_n \to 0$ in X and thereafter any $\lambda > 0$ is an eigenvalue of the problem (P_λ).*

Proof. The proof will be in three steps:

Step 1: The functional ψ_λ is even, $\psi_\lambda(0) = 0$, bounded from below and satisfies the (PS) condition.

It is clear that ψ_λ is even and $\psi_\lambda(0) = 0$.

According to the inequality of Hölder, we have for all $u \in X$

$$\int_\Omega b_1(x)|u|^{\beta(x)}dx \leq |b_1|_{s_1(x)} \left||u|^{\beta(x)}\right|_{s'(x)} \leq |b_1|_{s_1(x)}|u|^{\beta^i}_{s_1'(x)\beta(x)}, \tag{5}$$

where $i = -$ if $|u|_{s_1'(x)\beta(x)} < 1$ and $i = +$ if $|u|_{s_1'(x)\beta(x)} > 1$.

From the continuity of the injection $X \hookrightarrow L^{s_1'(x)\beta(x)}(\Omega)$, we can find a constant $c_3 > 0$ such that

$$|u|_{s_1'(x)\beta(x)} \leq c_3|u|, \forall u \in X. \tag{6}$$

Combining (5) and (6), we obtain

$$\int_\Omega b_1(x)|u|^{\beta(x)}dx \leq c_3^{\beta^i}|b_1|_{s_1(x)}\|u\|^{q^i}. \tag{7}$$

Therefore, from (7), we deduce that for any $u \in X$, we have

$$\begin{aligned}
\psi_\lambda(u) = &\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx + \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \int_\Omega \frac{a(x)}{\alpha(x)}|u|^{\alpha(x)}dx \\
&- \lambda\int_\Omega \frac{b_1(x)}{\beta(x)}|u|^{\beta(x)}dx + \lambda\int_\Omega \frac{b_2(x)}{\gamma(x)}|u|^{\gamma(x)}dx \\
&\geq \frac{1}{p^+}\Lambda_p(u) - \int_\Omega \frac{a(x)}{\alpha(x)}|u|^{\alpha(x)}dx - \lambda\int_\Omega \frac{b_1(x)}{\beta(x)}|u|^{\beta(x)}dx \\
&\geq \frac{1}{p^+}\eta(\|u\|) - \frac{a^+}{\alpha^-}\|u\|^{\alpha^i} - \frac{\lambda c_3^{\beta^i}}{\beta^-}|b_1|_{s_1(x)}\|u\|^{\beta^i},
\end{aligned}$$

where $\eta : [0, +\infty[\to \mathbb{R}$ is defined by

$$\eta(t) = \begin{cases} t^{p^+}, & \text{if } t \leq 1, \\ t^{p^-}, & \text{if } t > 1. \end{cases}$$

Since $\alpha^i < \beta^i < p^-$, then ψ_λ is bounded from below and coercive (because $\psi_\lambda(u) \to \infty$ when $\|u\| \to \infty$).

Let's show now that the functional ψ_λ satisfies the (PS) condition.

Let $(u_n) \subset X$ be a sequence of (PS), such that

$$\psi_\lambda(u_n) \text{ is bounded and } \psi_\lambda'(u_n) \to 0 \text{ in } X^*.$$

Then, by the coercivity of ψ_λ, the sequence (u_n) is bounded in X. According to the reflexivity of X, for a sub-sequence still noted (u_n), we have

$$u_n \rightharpoonup u \quad \text{in } X.$$

Let us then show that $u_n \to u$ in X.

First we have

$$\lim_{n\to\infty} \int_\Omega b_1(x)|u_n|^{\beta(x)-2}u_n(u_n - u)dx = 0 \tag{8}$$

Indeed, according to the inequality of Hölder, we have

$$\begin{aligned}\int_\Omega b_1(x)|u_n|^{\beta(x)-2}u_n(u_n-u)dx &\leq c|b_1(x)|_{s_1(x)}\left||u_n|^{\beta(x)-2}u_n\right|_{\frac{\beta(x)}{\beta(x)-1}}|u_n-u|_{\theta_1(x)}\ ,\ c>0\\ &\leq |b_1(x)|_{s_1(x)}(1+|u_n|^{\beta^+-1}_{\beta(x)})|u_n-u|_{\theta_1(x)}\end{aligned}$$

As X is continuously injected into $L^{\beta(x)}(\Omega)$ and $\{u_n\}_n^\infty$ is bounded in X, then $\{u_n\}_n^\infty$ is bounded in $L^{\beta(x)}(\Omega)$. On the other hand, since the injection $X \hookrightarrow L^{\theta_1(x)}(\Omega)$ is compact, we deduce that $|u_n-u|_{\theta_1(x)} \to 0$ when $n\to\infty$, which proves the relation (8). In the same way we get

$$\lim_{n\to\infty}\int_\Omega b_2(x)|u_n|^{\gamma(x)-2}u_n(u_n-u)dx = 0. \tag{9}$$

Similarly as above, by the following inequality

$$\left|\int_\Omega a(x)|u_n|^{\alpha(x)-2}u_n(u_n-u)dx\right| \leq 2a^+\left||u_n|^{\alpha(x)-2}u_n\right|_{\frac{\alpha(x)}{\alpha(x)-1}}|u_n-u|_{\alpha(x)}$$

we obtain

$$\lim_{n\to\infty}\int_\Omega a(x)|u_n|^{\alpha(x)-2}u_n(u_n-u)dx = 0. \tag{10}$$

On the other hand, we have

$$\lim_{n\to+\infty}\langle d\psi_\lambda(u_n), u_n-u\rangle = 0. \tag{11}$$

Indeed, we know that

$$\begin{aligned}|\langle d\psi_\lambda(u_n), u_n-u\rangle| &\leq |\langle d\psi_\lambda(u_n), u_n\rangle| + |\langle d\psi_\lambda(u_n), u\rangle|\\ &\leq \|d\psi_\lambda(u_n)\|_{X^*}\|u_n\| + \|d\psi_\lambda(u_n)\|_{X^*}\|u\|.\end{aligned}$$

Since $d\psi_\lambda(u_n)\to 0$ and $\{u_n\}_n^\infty$ is bounded in X, so this proves the relation (11).

Using (8), (9), (10) and (11), we deduce that

$$\lim sup\langle dF(u_n), u_n-u\rangle \leq 0,$$

with dF is the functional defined in Proposition III.5.

Since $u_n \rightharpoonup u$ in X and using the fact that dF satisfies the condition (S_+), then we deduce that $u_n\to u$ in X.

Step 2: For all $n\in\mathbb{N}^*$, there exists $H_n\in\Gamma_n$ such that

$$\sup_{u\in H_n}\psi_\lambda(u) < 0.$$

Indeed, let $b_1, b_2, \ldots, v_n \in C_0^\infty(\Omega)$ be such that $\mathrm{supp}(v_i)\cap\mathrm{supp}(v_j)=\emptyset$ if $i\neq j$
and $meas(\mathrm{supp}(v_j)) > 0$ for $i,j\in\{1,2,\ldots,n\}$ (where meas(.) is the Lebesgue measure of a set).

Consider $K_n = \mathrm{span}\{v_1, v_2, \ldots, v_n\}$. It is clear that $\dim K_n = n$ and for all $v\in K_n\setminus\{0\}$ one has

$$\int_\Omega a(x)|v(x)|^{\alpha(x)}dx > 0,\quad \int_\Omega b_1(x)|v(x)|^{\beta(x)}dx > 0\quad\text{and}\int_\Omega b_2(x)|v(x)|^{\gamma(x)}dx > 0.$$

Let us note $S=\{v\in X : \|v\|=1\}$ and $H_n(t) = t(S\cap K_n)$ for $0<t\leq 1$. Obviously, $\gamma(H_n(t)) = n$, for all $t\in]0,1]$.

Now, let us show that, for all $n\in\mathbb{N}\setminus\{0\}$, there exists $t_n\in]0,1]$ such that

$$\sup_{u\in H_n(t_n)}\psi_\lambda(u) < 0.$$

Indeed, for $0 < t \leq 1$, we have

$$\begin{aligned}
\sup_{u \in H_n(t)} \psi_\lambda(u) \leq & \sup_{v \in S \cap K_n} \psi_\lambda(tv) \\
&= \sup_{v \in S \cap K_n} \left\{ \int_\Omega \frac{t^{p(x)}}{p(x)} \left(|\Delta v(x)|^{p(x)} + |\nabla v(x)|^{p(x)} \right) dx \right. \\
&- \int_\Omega \frac{t^{\alpha(x)}}{\alpha(x)} a(x)|v(x)|^{\alpha(x)}\, dx - \lambda \int_\Omega \frac{t^{\beta(x)}}{\beta(x)} b_1(x)|v(x)|^{\beta(x)}\, dx \\
&\left. + \lambda \int_\Omega \frac{t^{\gamma(x)}}{\gamma(x)} b_2(x)|v(x)|^{\gamma(x)}\, dx \right\} \\
&\leq \sup_{v \in S \cap K_n} \left\{ \frac{t^{p^-}}{p^-} \Lambda_p(x) - \frac{t^{\alpha^+}}{\alpha^+} \int_\Omega a(x)|v(x)|^{\alpha(x)}\, dx \right. \\
&\left. - \frac{\lambda t^{\beta^+}}{\beta^+} \int_\Omega b_1(x)|v(x)|^{\beta(x)}\, dx + \frac{\lambda t^{\gamma^-}}{\gamma^-} \int_\Omega b_2(x)|v(x)|^{\gamma(x)}\, dx \right\} \\
&= \sup_{v \in S \cap K_n} \left\{ t^{p^-} \big(\frac{1}{p^-} - \frac{\lambda}{\beta^+} \frac{1}{t^{p^- - \beta^+}} \int_\Omega b_1(x)|v(x)|^{\beta(x)}\, dx \big) \right. \\
&\left. + t^{\gamma^-} \big(\frac{\lambda}{\gamma^-} \int_\Omega b_2(x)|v(x)|^{\gamma(x)}\, dx - \frac{1}{\alpha^+} \frac{1}{t^{\gamma^- - \alpha^+}} \int_\Omega a(x)|v(x)|^{\alpha(x)}\, dx \big) \right\}.
\end{aligned}$$

Since $m_1 := \min_{v \in S \cap K_n} \int_\Omega b_1(x)|v(x)|^{\beta(x)}\, dx > 0$, $m_2 := \max_{v \in S \cap K_n} \int_\Omega b_2(x)|v(x)|^{\gamma(x)}\, dx > 0$ and $m_3 := \min_{v \in S \cap K_n} \int_\Omega a(x)|v(x)|^{\alpha(x)}\, dx > 0$ we can choose $t_n \in]0, 1]$ which is small enough such that

$$\frac{1}{p^-} - \frac{\lambda}{q^+} \frac{1}{t_n^{p^- - q^+}} m_1 < 0,$$

and

$$\frac{\lambda}{\gamma^-} m_2 - \frac{1}{\alpha^+} \frac{1}{t_n^{\gamma^- - \alpha^+}} m_3 < 0.$$

Consequently

$$\sup_{u \in H_n(t_n)} \psi_\lambda(u) < 0.$$

Step 3: Since all assumptions of the symmetric mountain pass lemma are verified, then each $c_k := \inf_{H \in \Gamma_k} \sup_{u \in H} \psi_\lambda(u)$ is a critical value of $\psi_\lambda(u)$, and the latter admits a sequence of non-trivial weak solutions $(u_n)_n$ such that for all n, we have

$$u_n \neq 0, \quad \psi'_\lambda(u_n) = 0, \quad \psi_\lambda(u_n) \leq 0, \quad \lim_n u_n = 0.$$

□

IV. Final comments

We have to mention that in the particular case when $b_2(x) \equiv a(x) \equiv 0$, we find that our problem is identical to the problem studied in [19], this shows that our main result constitutes an important development of the main result of [19], noting that in the latter the Ljusternik-Schnireleman principle on C^1-manifold was employed in the proof.

In the same context, when $b_2(x) \equiv a(x) \equiv 0$ and we have only the p(x)-biharmonic operator, the equation of problem (P_λ) reduces to the nonhomogeneous elliptic equation with variable exponent

$$\Delta^2_{p(x)} = \lambda b_1(x)|u|^{\beta(x)-2}u,$$

which is the same that one studied in [20] and [21]. Therefore, Theorem III.7 obtained in our paper is a natural extension to Theorem 3.1 in [20]. Notice that the latter is proved using Ekeland's variational principle.

Finally, we mention that in the case of $\alpha(x) = p(x)$ and we have only the p(x)-biharmonic operator, our main result is complementary with the result obtained in [7] and [8] with one difference that in the latter it was assumed that a is a negative function while in our article we assumed the opposite.

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