Enumeration of Triangles and Hamiltonian Property of The Zero-Divisor Cayley Graph of The Ring $(Z_n,\oplus,\odot)$

Jangiti Devendra, Levaku Madhavi, and Tippaluri Nagalakshumma

***Abstract* — In this paper an enumeration method to find the number of triangles in the zero-divisor Cayley graph $G(Z_n, D_0)$, associated with the ring $(Z_n, \oplus, \odot), n \geq 1$, an integer and its subset D_0 of zero-divisors is presented. Further it is shown that this graph is Hamiltonian, not bipartite and Eulerian, when n is odd.**

***Keywords* — Zero-divisor, Zero-divisor Cayley graph, Triangle, Basic triangle and Hamilton cycle.**
AMS Subject Classification: 05C30, 05C38, 05C45, 68R05

I. Introduction

Reference [1] introduced the concept of congruence in Graph theory and thus paved the way for the emergence of a new class of graphs, namely, Arithmetic graphs. Reference [2], and [3], [4] and others studied the Cayley graphs associated with certain arithmetic functions. Reference [5] introduced Cayley graphs associated with the arithmetical functions, namely, the Euler totient function $\varphi(n)$, the quadratic residues modulo a prime p and the divisor function $d(n), n \geq 1$, an integer and obtained various properties of these graphs. Later [6]-[8], studied various domination parameters and cycle structures of these graphs.

References [9]-[13], and others studied the zero-divisor graphs of commutative rings. Given a commutative ring R with identity, they defined the zero-divisor graph $\Gamma(R)$ as the graph, whose vertex set is the set $Z(R)^*$, the set of nonzero zero-divisors of R and the edge set is the set of all ordered pairs (x, y) of elements $x, y \in Z(R)^*$, such that $xy = 0$ and studied the connectedness, the girth, the diameter, the automorphism of $\Gamma(R)$ and other properties under conditions on the ring R. In [14]-[16], the authors introduced Cayley graphs associated with the set of zero-divisors of a ring $(R, +, \cdot)$ and studied these graphs, with particular reference to the ring $(Z_n, \oplus, \odot)$ of residue classes modulo $n \geq 1$, an integer and studied their basic properties, vertex domination, girth, diameter, and other concepts.

Consider the ring $(Z_n, \oplus, \odot)$ of integers modulo n, $n \geq 1$, an integer, which is a commutative ring with unity. In [15], it is established that the set D_0 of nonzero zero-divisors in the ring $(Z_n, \oplus, \odot)$ is a symmetric subset of the group $(Z_n, \oplus)$ and the zero-divisor Cayley graph $G(Z_n, D_0)$ is the graph, whose vertex set is Z_n and the edge set is the set of ordered pairs (u, v)such that $u, v \in Z_n$ and either $u - v \in D_0$ or $v - u \in D_0$. This graph is $(n - \varphi(n) - 1) -$ regular and its size is $\frac{n}{2}(n - \varphi(n) - 1)$. Further $G(Z_n, D_0)$ contains (i) only isolated vertices, if n is a prime (Lemma 2.10, [15]), (ii) p disjoint complete components, if $n = p^r, p$, a prime and $r > 1$, an integer (Theorem 3.7, [15]), (iii) it is a connected graph, if $n > 1$, an integer, which is not a power of a single prime (Theorem 4.4, [15]). The terminology and notions that are used in this paper can be found in [17] for graph theory, [18] for algebra and [19] for number theory.
The graphs $G(Z_7, D_0), G(Z_9, D_0)$ and $G(Z_{12}, D_0)$ are given below:

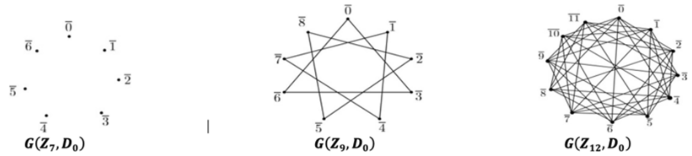

Fig. 1. The graphs of $G(Z_7, D_0), G(Z_9, D_0)$ and $G(Z_{12}, D_0)$

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II. Enumeration of Triangles in A Zero-Divisor Cayley Graph

In a graph a 3-cycle is called a **triangle** and the triangle formed by the distinct vertices a, b, c is denoted by (a, b, c). In the graph $G(Z_n, D_0)$, the triangle $(\bar{0}, \bar{a}, \bar{b})$ is called a **basic triangle** and it is denoted by $\Delta_{\bar{a},\bar{b}}$.

***Lemma 2.1*:** The number of distinct basic triangles in the graph $G(Z_n, D_0)$ is $\frac{1}{2}\sum_{\bar{a}\in D_0}|D_0 \cap (\bar{a} + D_0)|$.

***Proof*:** Let $\bar{a} \in D_0$ and let $\bar{b} \in Z_n$. Then $(\bar{0}, \bar{a}, \bar{b})$ is a basic triangle

$\Leftrightarrow \bar{a} - \bar{0} \in D_0, \bar{b} - \bar{0} \in D_0$ and $\bar{b} - \bar{a} \in D_0$

$\Leftrightarrow \bar{a}, \bar{b}, \bar{b} - \bar{a} \in D_0$

$\Leftrightarrow \bar{a}, \bar{b} \in D_0$ and $\bar{b} \in \bar{a} + D_0$

$\Leftrightarrow \bar{b} \in D_0 \cap (\bar{a} + D_0)$.

Thus, for $\bar{a} \in D_0$, the number of distinct basic triangles in the graph $G(Z_n, D_0)$ is $|D_0 \cap (\bar{a} + D_0)|$. However by the definition of the adjacency in $G(Z_n, D_0)$, the triangles $\Delta_{\bar{a},\bar{b}}$ and $\Delta_{\bar{b},\bar{a}}$ represent the same basic triangle. So the number of distinct basic triangles in the graph $G(Z_n, D_0)$ is:

$$\frac{1}{2}\sum_{\bar{a}\in D_0}|D_0 \cap (\bar{a} + D_0)|. \qquad \blacksquare$$

Example 2.2: The basic triangles of the zero-divisor Cayley graph $G(Z_{12}, D_0)$ are enumerated below. Here $D_0 = \{\bar{2}, \bar{3}, \bar{4}, \bar{6}, \bar{8}, \bar{9}, \overline{10}\}$.

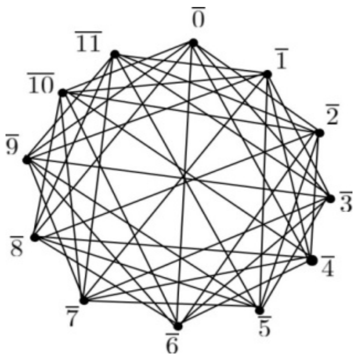

Fig. 2. The graph $G(Z_{12}, D_0)$.

TABLE I: Enumeration of Basic Triangles of $G(Z_{12}, D_0)$

$\bar{a}$	$\bar{a}+D_0$	$D_0 \cap (\bar{a} + D_0)$	Basic Tringles
$\bar{2}$	$\{\bar{4}, \bar{5}, \bar{6}, \bar{8}, \overline{10}, \overline{11}, \bar{0}\}$	$\{\bar{4}, \bar{6}, \bar{8}, \overline{10}\}$	$(\bar{0}, \bar{2}, \bar{4})$,$(\bar{0}, \bar{2}, \bar{6})$,$(\bar{0}, \bar{2}, \bar{8})$,$(\bar{0}, \bar{2}, \overline{10})$
$\bar{3}$	$\{\bar{5}, \bar{6}, \bar{7}, \bar{9}, \overline{11}, \bar{0}, \bar{1}\}$	$\{\bar{6}, \bar{9}\}$	$(\bar{0}, \bar{3}, \bar{6})$, $(\bar{0}, \bar{3}, \bar{9})$
$\bar{4}$	$\{\bar{6}, \bar{7}, \bar{8}, \overline{10}, \bar{0}, \bar{1}, \bar{2}\}$	$\{\bar{2}, \bar{6}, \bar{8}, \overline{10}\}$	$(\bar{0}, \bar{4}, \bar{2})$,$(\bar{0}, \bar{4}, \bar{6})$, $(\bar{0}, \bar{4}, \bar{8})$, $(\bar{0}, \bar{4}, \overline{10})$
$\bar{6}$	$\{\bar{8}, \bar{9}, \overline{10}, \bar{0}, \bar{2}, \bar{3}, \bar{4}\}$	$\{\bar{2}, \bar{3}, \bar{4}, \bar{8}, \bar{9}, \overline{10}\}$	$(\bar{0}, \bar{6}, \bar{2})$, $(\bar{0}, \bar{6}, \bar{4})$, $(\bar{0}, \bar{6}, \bar{8})$, $(\bar{0}, \bar{6}, \overline{10})$
$\bar{8}$	$\{\overline{10}, \overline{11}, \bar{0}, \bar{2}, \bar{4}, \bar{5}, \bar{6}\}$	$\{\bar{2}, \bar{4}, \bar{6}, \overline{10}\}$	$(\bar{0}, \bar{8}, \bar{2})$, $(\bar{0}, \bar{8}, \bar{4})$$(\bar{0}, \bar{8}, \bar{6})$, $(\bar{0}, \bar{8}, \overline{10})$
$\bar{9}$	$\{\overline{11}, \bar{0}, \bar{1}, \bar{3}, \bar{5}, \bar{6}, \bar{7}\}$	$\{\bar{3}, \bar{6}\}$	$(\bar{0}, \bar{9}, \bar{3})$, $(\bar{0}, \bar{9}, \bar{6})$
$\overline{10}$	$\{\bar{0}, \bar{1}, \bar{2}, \bar{4}, \bar{6}, \overline{11}, \bar{8}\}$	$\{\bar{2}, \bar{4}, \bar{6}, \bar{8}\}$	$(\bar{0}, \overline{10}, \bar{2})$, $(\bar{0}, \overline{10}, \bar{4})$$(\bar{0}, \overline{10}, \bar{8})$

Since $\Delta_{\bar{a},\bar{b}} = \Delta_{\bar{b},\bar{a}}$, for $\bar{a}, \bar{b} \in D_0$, the distinct basic triangles of the graph $G(Z_{12}, D_0)$ are $\Delta_{\bar{2},\bar{4}}$, $\Delta_{\bar{2},\bar{6}}$, $\Delta_{\bar{2},\bar{8}}$, $\Delta_{\bar{2},\overline{10}}$, $\Delta_{\bar{3},\bar{6}}$, $\Delta_{\bar{3},\bar{9}}$, $\Delta_{\bar{4},\bar{6}}$, $\Delta_{\bar{4},\bar{8}}$, $\Delta_{\bar{4},\overline{10}}$, $\Delta_{\bar{6},\bar{8}}$, $\Delta_{\bar{6},\overline{10}}$ and $\Delta_{\bar{8},\overline{10}}$, which are represented by the thick lines in the graph $G(Z_{12}, D_0)$ and the extracted portion of the graph containing, the basic triangles is given in the Fig. 4.

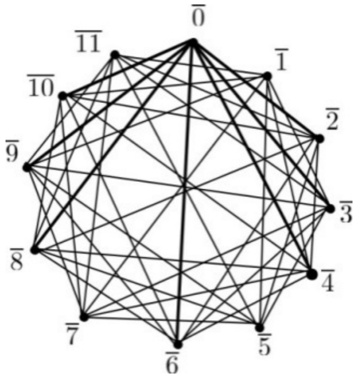

Fig. 3. The graph $G(Z_{12}, D_0)$.

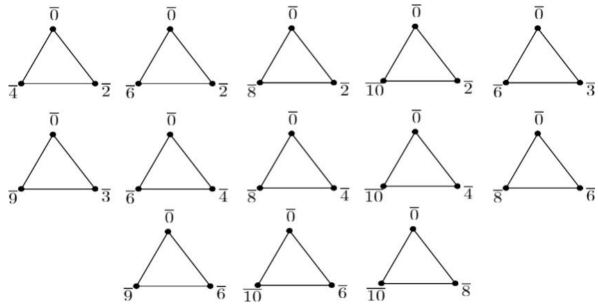

Fig. 4. The basic triangles of $G(Z_{12}, D_0)$.

The following theorem gives a formula for the total number of distinct basic triangles in the graph $G(Z_n, D_0)$.

Theorem 2.3: The number of distinct triangles in the graph $G(Z_n, D_0)$ is:

$$\frac{1}{6}|Z_n| \sum_{\bar{a}\in D_0} |D_0 \cap (\bar{a} + D_0)|.$$

Proof: The graph $G(Z_n, D_0)$, being a Cayley graph, it is vertex symmetric. Further, for any $\bar{g} \in G(Z_n, D_0)$, the mapping: $G(Z_n, D_0) \to G(Z_n, D_0)$, given by $\theta(\bar{x}) = \bar{g} + \bar{x}$, for all $G(Z_n, D_0)$, is an automorphism of $G(Z_n, D_0)$ and $\theta(\bar{0}) = \bar{g} + \bar{0} = \bar{g}$. Since an automorphism preserves incidence, it takes adjacent vertices into adjacent vertices and non-adjacent vertices into non-adjacent vertices and the basic triangle $(\bar{0}, \bar{a}, \bar{b})$ is taken into the triangle $(\bar{g}, \bar{g} + \bar{a}, \bar{g} + \bar{b})$ under the automorphism θ. Thus for each $\bar{g} \in G$, the number of triangles of the form $(\bar{g}, \bar{x}, \bar{y})$ is also $\frac{1}{2}\sum_{\bar{a}\in D_0}|D_0 \cap (\bar{a} + D_0)|$ and the total number of triangles in $G(Z_n, D_0)$ is given by $\frac{1}{2}|Z_n| \sum_{\bar{a}\in D_0}|D_0 \cap (\bar{a} + D_0)|$.

However in the above enumeration each triangle in $G(Z_n, D_0)$ is counted thrice, namely, once by each of its three vertices. So the total number of distinct triangles in $G(Z_n, D_0)$ is:

$$\frac{1}{6}|Z_n| \sum_{\bar{a}\in D_0} |D_0 \cap (\bar{a} + D_0)|.$$ ■

Example 2.4: For the graph $G(Z_6, D_0)$, $Z_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ and $D_0 = \{\bar{2}, \bar{3}, \bar{4}\}$.
The sets $\bar{a} + D_0$, $D_0 \cap (\bar{a} + D_0)$ and $|D_0 \cap (\bar{a} + D_0)|$ are given in the Table II.

TABLE II: ENUMERATION OF DISTINCT TRIANGLES OF $G(Z_6, D_0)$

$\bar{a}$	$\bar{a} + D_0$	$D_0 \cap (\bar{a} + D_0)$	$\lvert D_0 \cap (\bar{a} + D_0)\rvert$
$\bar{2}$	$\{\bar{4}, \bar{5}, \bar{0}\}$	$\{\bar{4}\}$	1
$\bar{3}$	$\{\bar{5}, \bar{0}, \bar{1}\}$	$\{\emptyset\}$	0
$\bar{4}$	$\{\bar{0}, \bar{1}, \bar{2}\}$	$\{\bar{2}\}$	1

Hence $G(Z_6, D_0)$ contains $\frac{1}{6} \times 6|1 + 0 + 1| = 2$ distinct triangles, which are given below.

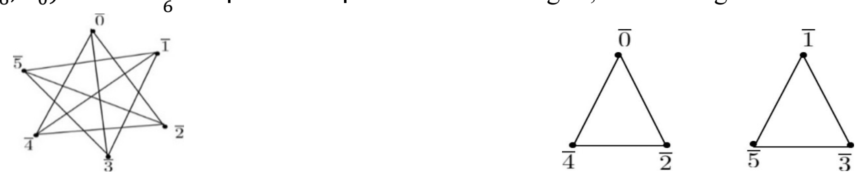

Fig. 5. The graph $G(Z_6, D_0)$ and its triangles.

Example 2.5: Consider the zero-divisor Cayley graph $G(Z_{12}, D_0)$. Here $Z_{12} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}, \overline{10}, \overline{11}\}$ and $D_0 = \{\bar{2}, \bar{3}, \bar{4}, \bar{6}, \bar{8}, \bar{9}, \overline{10},\}$.
The sets $\bar{a} + D_0$, $D_0 \cap (\bar{a} + D_0)$ and $|D_0 \cap (\bar{a} + D_0)|$ are given in the Table III.

TABLE III: ENUMERATION OF DISTINCT TRIANGLES OF $G(Z_{12}, D_0)$

$\bar{a}$	$\bar{a} + D_0$	$D_0 \cap (\bar{a} + D_0)$	$\lvert D_0 \cap (\bar{a} + D_0)\rvert$
$\bar{2}$	$\{\bar{4}, \bar{5}, \bar{6}, \bar{8}, \overline{10}, \overline{11}, \bar{0}\}$	$\{\bar{4}, \bar{6}, \bar{8}, \overline{10}\}$	4
$\bar{3}$	$\{\bar{5}, \bar{6}, \bar{7}, \bar{9}, \overline{11}, \bar{0}, \bar{1}\}$	$\{\bar{6}, \bar{9}\}$	2
$\bar{4}$	$\{\bar{6}, \bar{7}, \bar{8}, \overline{10}, \bar{1}, \bar{2}\}$	$\{\bar{2}, \bar{6}, \bar{8}, \overline{10}\}$	4
$\bar{6}$	$\{\bar{8}, \bar{9}, \overline{10}, \bar{0}, \bar{2}, \bar{4}\}$	$\{\bar{2}, \bar{3}, \bar{4}, \bar{8}, \bar{9}, \overline{10}\}$	6
$\bar{8}$	$\{\overline{10}, \overline{11}, \bar{0}, \bar{2}, \bar{4}, \bar{5}, \bar{6}\}$	$\{\bar{2}, \bar{4}, \bar{6}, \overline{10},\}$	4
$\bar{9}$	$\{\overline{11}, \bar{0}, \bar{1}, \bar{3}, \bar{5}, \bar{6}, \bar{7}\}$	$\{\bar{3}, \bar{6}\}$	2
$\overline{10}$	$\{\bar{0}, \bar{1}, \bar{2}, \bar{4}, \bar{6}, \bar{7}, \bar{8}\}$	$\{\bar{2}, \bar{4}, \bar{6}, \bar{8}\}$	4

Hence the number $T(G(Z_{12}, D_0))$ of distinct triangles of $G(Z_n, D_0)$ is given by

$$T(G(Z_{12}, D_0)) = \frac{1}{6}|Z_n| \sum_{\bar{a}\in D_0} |D_0 \cap (\bar{a} + D_0)| = \frac{12}{6}|4 + 2 + 4 + 6 + 4 + 2 + 4| = 2(26) = 52.$$

These 52 distinct triangles of $G(Z_{12}, D_0)$ are given below:
$(\bar{0},\bar{2},\bar{4})$, $(\bar{0},\bar{2},\bar{6})$, $(\bar{0},\bar{2},\bar{8})$, $(\bar{0},\bar{2},\overline{10})$, $(\bar{0},\bar{3},\bar{6})$, $(\bar{0},\bar{3},\bar{9})$, $(\bar{0},\bar{4},\bar{6})$, $(\bar{0},\bar{4},\bar{8})$, $(\bar{0},\bar{4},\overline{10})$, $(\bar{0},\bar{6},\bar{8})$, $(\bar{0},\bar{6},\bar{9})$, $(\bar{0},\bar{6},\overline{10})$, $(\bar{0},\bar{8},\overline{10})$, $(\bar{1},\bar{3},\bar{5})$, $(\bar{1},\bar{3},\bar{7})$, $(\bar{1},\bar{3},\bar{9})$, $(\bar{1},\bar{3},\overline{11})$, $(\bar{1},\bar{4},\bar{7})$, $(\bar{1},\bar{4},\overline{10})$, $(\bar{1},\bar{5},\bar{7})$, $(\bar{1},\bar{5},\bar{9})$, $(\bar{1},\bar{5},\overline{11})$, $(\bar{1},\bar{7},\bar{9})$, $(\bar{1},\bar{7},\overline{10})$, $(\bar{1},\bar{7},\overline{11})$, $(\bar{1},\bar{9},\overline{11})$, $(\bar{2},\bar{4},\bar{6})(\bar{2},\bar{4},\bar{8})$, $(\bar{2},\bar{4},\overline{10})$, $(\bar{2},\bar{5},\bar{8})$, $(\bar{2},\bar{5},\overline{11})$, $(\bar{2},\bar{6},\bar{8})$, $(\bar{2},\bar{6},\overline{10})$, $(\bar{2},\bar{8},\overline{10})$, $(\bar{2},\bar{8},\overline{11})$, $(\bar{3},\bar{5},\bar{7})$, $(\bar{3},\bar{5},\bar{9})$, $(\bar{3},\bar{5},\overline{11})$, $(\bar{3},\bar{6},\bar{9})$, $(\bar{3},\bar{7},\bar{9})$, $(\bar{3},\bar{7},\overline{11})$, $(\bar{3},\bar{9},\overline{11})$, $(\bar{4},\bar{6},\bar{8})$, $(\bar{4},\bar{6},\overline{10})$, $(\bar{4},\bar{7},\overline{10})$, $(\bar{4},\bar{8},\overline{10})$, $(\bar{5},\bar{7},\bar{9})$, $(\bar{5},\bar{7},\overline{11})$, $(\bar{5},\bar{8},\overline{11})$, $(\bar{5},\bar{9},\overline{11})$, $(\bar{6},\bar{8},\overline{10})$, $(\bar{7},\bar{9},\overline{11})$.

III. Hamiltonian Property of the Zero-Divisor Cayley Graph $G(Z_n, D_0)$

A **Hamilton cycle** of G is a cycle that contains every vertex of G exactly once. A graph is called **Hamiltonian** if it contains a Hamilton cycle.

Theorem 3.1: If $n > 1$, be an integer, not a power of a single prime, then the graph $G(Z_n, D_0)$ is connected and Hamiltonian.

Proof: Let $n > 1$ be an integer, which is not a power of a single prime. By the Theorem 4.4, of [15], the graph $G(Z_n, D_0)$ is connected. The vertex set V of $G(Z_n, D_0)$ can be viewed as the disjoint union of the subsets $V_0, V_1, V_2, \dots, V_{p_1-1}$ of V, where:

$$V_0 = \left\{0\overline{p_1}, 1\overline{p_1}, 2\overline{p_1}, \dots, i\overline{p_1}, \dots, \left(\frac{n-p_1}{p_1}\right)\overline{p_1}\right\},$$
$$V_1 = \left\{0\overline{p_1} + \overline{p_2}, 1\overline{p_1} + \overline{p_2}, 2\overline{p_1} + \overline{p_2}, \dots, i\overline{p_1} + \overline{p_2}, \dots, \left(\frac{n-p_1}{p_1}\right)\overline{p_1} + \overline{p_2}\right\},$$
$$\vdots$$
$$V_{p_1-1} = \left\{0\overline{p_1} + (p_1-1)\overline{p_2}, \dots, i\overline{p_1} + (p_1-1)\overline{p_2}, \dots, \left(\frac{n-p_1}{p_1}\right)\overline{p_1} + (p_1-1)\overline{p_2}\right\}.$$

Let us arrange the vertices of $V_0, V_1, V_2, \dots, V_{p_1-1}$ in the sequence given below, starting with $\bar{0}$ and ending with $\bar{0}$, namely,

$$H = \left(\bar{0}, \overline{p_1}, \dots \overline{ip_1}, \overline{p_1\left(\frac{n-p_1}{p_1}\right)}, \overline{\left(\frac{n-p_1}{p_1}\right)p_1} + \overline{p_2}, \dots, \overline{p_1} + \overline{p_2}, \overline{p_2}, \dots, \overline{p_1\left(\frac{n-p_1}{p_1}\right)} + 2\overline{p_2}, \dots, \overline{(p_1-1)p_2}, \bar{0}\right).$$

This fact is elegantly exhibited in the following array with the cycle indicated by directed arrows.

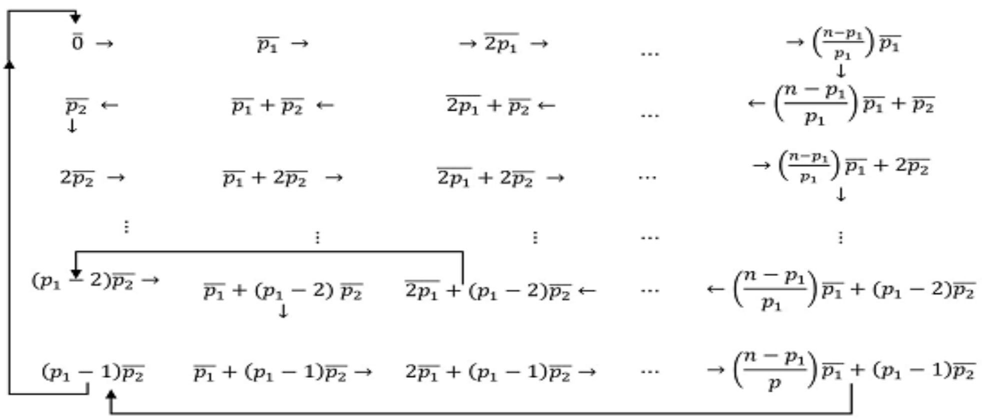

Fig. 6. Array indicating a Hamilton Cycle of $G(Z_n, D_0)$

One can see that all the vertices of $G(Z_n, D_0)$ occur exactly once in the above sequence. Further, it is easy to verify that there is an edge between any two consecutive vertices in the above sequence. For example $\left((i+1)\overline{p_1} + \overline{p_2}\right) - \left(i\overline{p_1} + \overline{p_2}\right) = \overline{p_1}$, which is a zero-divisor in the ring $(Z_n, \oplus, \odot)$, so that, there is an edge between these two consecutive vertices in the sequence. Hence the cycle H formed by the above sequence of vertices of $G(Z_n, D_0)$ is a Hamilton cycle and the graph $G(Z_n, D_0)$ is Hamiltonian. ■

Corollary 3.2: If $n > 1$, is an integer, not a prime, then the graph $G(Z_n, D_0)$ is not bipartite.

Proof: The zero-divisor Cayley graph $G(Z_n, D_0)$ has a triangle $(\overline{0}, \overline{p_1}, \overline{2p_1})$, which is of length three, when $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, where $p_1 < p_2 < \cdots < p_r$ are primes, $\alpha_i \geq 1$ and $r > 1$ are integers. So that the zero divisor Cayley graph $G(Z_n, D_0)$ is not bipartite, since a bipartite graph contains no odd cycles . ■

Theorem 3.3: If $n > 1$, is an odd integer, which is not a prime, then the graph $G(Z_n, D_0)$ is Eulerian.

Proof: Let $n > 1$, be an odd integer, which is not a prime. By the Lemma 2.6 of [15], the degree of each vertex in $G(Z_n, D_0)$ is $n - \varphi(n) - 1$. Since n is odd and $\varphi(n)$ is even ($n \geq 3$), it follows that, $n - \varphi(n) - 1$ is even. Since a connected graph is Eulerian, if, and only if, the degree of each of its vertex is even Theorem 4.1 of [17], it follows that the graph $G(Z_n, D_0)$ is Eulerian. ■

Example 3.4: Consider the graph $G(Z_{35}, D_0)$. Here $35 = 5 \times 7, p_1 = 5$ and $p_2 = 7$. As in the Theorem 3.7 of [15], the vertex set is a disjoint union of V_0, V_1, V_2, V_3 and V_4, where $V_0 = \{\overline{0}, \overline{5}, \overline{10}, \overline{15}, \overline{20}, \overline{25}, \overline{30}\}$, $V_1 = \{\overline{7}, \overline{12}, \overline{17}, \overline{22}, \overline{27}, \overline{32}, \overline{2}\}$, $V_2 = \{\overline{14}, \overline{19}, \overline{24}, \overline{29}, \overline{34}, \overline{4}, \overline{9}\}$, $V_3 = \{\overline{21}, \overline{26}, \overline{31}, \overline{1}, \overline{6}, \overline{11}, \overline{16}\}$ and $V_4 = \{\overline{28}, \overline{3}, \overline{8}, \overline{13}, \overline{18}, \overline{23}, \overline{31}\}$. Hence

$$(\overline{0}, \overline{5}, \overline{10}, \overline{15}, \overline{20}, \overline{25}, \overline{30}, \overline{32}, \overline{27}, \overline{22}, \overline{17}, \overline{12}, \overline{7}, \overline{14}, \overline{19}, \overline{24}, \overline{29}, \overline{34}, \overline{4}, \overline{9}, \overline{16}, \overline{11}, \overline{6}, \overline{1}, \overline{31}, \overline{21}, \overline{26}, \overline{3}, \overline{8}, \overline{13}, \overline{18}, \overline{23}, \overline{28}, \overline{0})$$

is a Hamilton cycle in the graph $G(Z_{35}, D_0)$. This Hamilton cycle is exhibited by boldface edges and this Hamilton cycle is extracted from the graph $G(Z_{35}, D_0)$ and it is shown below:

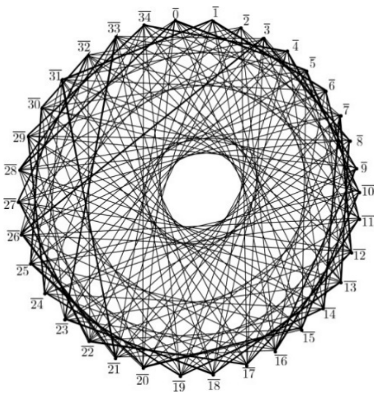

Fig. 7. The graph $G(Z_{35}, D_0)$.

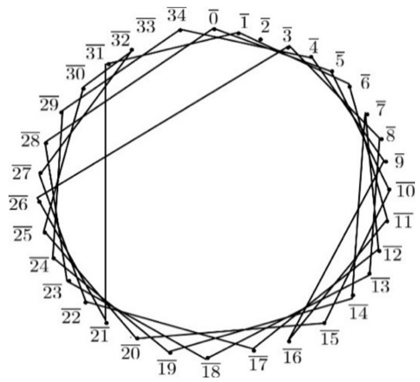

Fig. 8. The Hamilton cycle of $G(Z_{35}, D_0)$.

IV. Conclusion

Reference [6] gave a method of enumeration of disjoint Hamilton cycles in a divisor Cayley graph. We wish to obtain an enumeration method, which gives the number of Hamilton cycles in the zero-divisor Cayley graph $G(Z_n, D_0)$. Suppose $n > 5$, is an even integer, which is not a power of a single prime. Then by the Lemma 2.1 of [15], the degree of each vertex is $n - \varphi(n) - 1$ and its size is $\frac{n(n-\varphi(n)-1)}{2}$. Since n is even and $\varphi(n)$ is even for $n \geq 3$, it follows that $n - \varphi(n) - 1$ is odd and thus degree of each vertex of the graph $G(Z_n, D_0)$ is odd. If the graph $G(Z_n, D_0)$ is decomposed into the union of k disjoint Hamilton cycles, each of which contains n edges, the number of edges in the graph $G(Z_n, D_0)$ is nk, so that $nk = \frac{n(n-\varphi(n)-1)}{2}$, or, $k = \frac{(n-\varphi(n)-1)}{2}$. This is not possible since $(n - \varphi(n) - 1)$ is odd. So the graph $G(Z_n, D_0)$ cannot be decomposed into edge disjoint Hamilton cycles. However if n is odd but not a power of a single prime, similar argument shows that it may be possible that the graph can be decomposed into edge disjoint Hamilton cycles. In this case, it will be interesting to find an enumeration method to find the number of edge disjoint Hamilton cycles of the graph $G(Z_n, D_0)$.

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CONFLICT OF INTEREST

Authors declare that they do not have any conflict of interest.

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